- 1. Let $X_i \overset{iid}{\sim} \mathcal{U}(0,\theta)$ for $i = 1, ..., n$.

  (a) Show that $X_{(n)}/\theta$ is a pivotal quantity.
  Let $Y_i = X_i/\theta$. We know (previous hws) that $Y_i \sim U(0,1)$. Then $X_{(n)}/\theta = Y_{(n)}$ has a distribution independent of θ, and thus it is a pivotal quantity.

  (b) Derive a formula for a $100(1-\alpha)\%$ confidence interval for θ.

$$
\begin{aligned}
P_\theta(\theta \in [aX_{(n)}, bX_{(n)}]) &= P_\theta(aX_{(n)} \le \theta \le bX_{(n)}) \\
&= P_\theta(1/b \le Y_{(n)} \le 1/a) \\
&= \int_{1/b}^{1/a} nt^{n-1} dt = (1/a)^n - (1/b)^n
\end{aligned}
$$

  So a $100(1-\alpha)\%$ CI would be $[aX_{(n)}, bX_{(n)}]$ that satisfies $a, b > 0$ and $(1/a)^n - (1/b)^n = 1-\alpha$.
  Note: Practically, it wouldn't make sense to have a lower and upper bound in this case since it is impossible to have the maxiumum value be greater than θ. Thus, a one-sided confidence interval would make more sense, but a two sided one is presented for an example.

  (c) Find the expected width of your confidence interval for θ.

$$
\begin{aligned}
E_\theta(bX_{(n)} - aX_{(n)}) &= (b-a)E_\theta(X_{(n)}) \\
&= (b-a)(\frac{n-1}{n})\theta
\end{aligned}
$$

- 2. Let $X_i \overset{iid}{\sim} \mathcal{B}(1,p)$ for $i = 1, ..., n$. Suppose we observe $\sum x_i = 0$. Find a 95% upper confidence bound for p. Show that for large n, this bound is approximately 3/n.

$$
\begin{aligned}
P_{p_U}(\sum X_i \le 0) &= (1-p_U)^n = 0.05 \\
\Rightarrow p_U &= 1 - 0.05^{1/n}
\end{aligned}
$$

For large n, assume $\sum X_i \dot\sim N(np, npq)$

$$
P_{p_U}(\sum X_i \le 0) = P_{p_U}\left(\frac{\sum X_i - np}{(npq)^{1/2}} \le -(np/q)^{1/2}\right)
$$

$$\begin{aligned}
(np/q)^{1/2} &\approx 1.64 \\
p &\approx 2.7(1-p)/n \\
p &\approx 2.7/n(1-2.7/n) \\
p &\approx 2.7/(n-2.7) \\
p &\approx 3/n \text{ As n gets large}
\end{aligned}$$

Something else one might consider is using a Taylor's series expansion after taking the ln of both sides since $\ln(1-p) \approx -p$ and $\ln(0.05) = -2.9975$.

- 3. Suppose $X_i, i = 1, ..., n$ are independent and identically distributed random variables which, conditional upon a parameter $\theta > 0$, have the exponential distribution $\mathcal{E}(\theta)$ with density $f(x) = \theta e^{-\theta x}, x > 0$, and 0 otherwise.
  Consider a prior distribution for θ according to the gamma distribution $\theta \sim \Gamma(\alpha, \beta)$ with density $b(\theta) = \beta^\alpha \theta^{\alpha-1} e^{-\beta\theta}/\Gamma(\alpha)$ and mean α/β

  (a) Show that the above prior distribution is the conjugate prior for this problem.
  The prior will be conjugate if the resulting distribution is also gamma.

$$\begin{aligned}
p(\theta|\vec{x}) &\propto p(\vec{x}\,|\theta)\ \pi(\theta) = \prod p(x_i|\theta)\ \pi(\theta) \\
&= \prod_{i=1}^{n} \left(\theta e^{-\theta x_i}\right) \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\theta\beta} \\
&\propto \theta^n e^{-\theta \sum x_i} \theta^{\alpha-1} e^{-\theta\beta} = \theta^{\alpha+n-1} e^{-\theta(\beta+\sum x_i)} \\
&\propto \Gamma(\alpha+n, \beta+\sum x_i)
\end{aligned}$$

  Thus gamma is conjugate for the exponential since the prior was a gamma as was the posterior.

  (b) Find the posterior distribution of $\theta|(X_1, ..., X_n)$.
  Done in part (a).

  (c) Find the Bayes estimator for squared error loss, i.e. $L(\theta, d) = (\theta - d)^2$.
  The Bayes estimator to minimize squared error loss is the posterior mean.

$$E(\theta|X) = E\left[\Gamma(\alpha+n, \beta+\sum x_i)\right] = \frac{\alpha+n}{\beta+\sum x_i}$$

(d) Consider now the case of observing a single additional random variable X_{n+1} which is independent of the previous sample and distributed according to $X_{n+1}|\theta \sim \mathcal{E}(\theta)$. Using the posterior distribution found in (b) as your prior, find the posterior distribution of θ based on the observation of X_{n+1}.

$$\begin{aligned}
p(\theta|x_{n+1}) &\propto p(x_{n+1}|\theta)\ \pi^*(\theta) \\
&= \theta e^{-\theta x_{n+1}} \frac{(\beta + \sum x_i)^{\alpha+n}}{\Gamma(\alpha+n)} \theta^{\alpha+n-1} e^{-\theta(\beta+\sum x_i)} \\
&\propto \theta^{\alpha+n+1-1} e^{-\theta(\beta+\sum x_i + x_{n+1})} \\
&\propto \Gamma(\alpha+n+1, \beta+\sum x_i + x_{n+1})
\end{aligned}$$

Note: This is the same result as if we started with all $i = 1, ..., n+1$ data points.

(e) Compare the posterior distribution of θ derived in (d) to that obtained by using the original prior ($\theta \sim \Gamma(\alpha, \beta)$) and the total sample having n + 1 observations $X_1, ..., X_n, X_{n+1}$. It's the same as noted in part (d).