Stat 513
HW 6 Solutions
March 7, 2004

- 9.3.1

(a) Let $X_1, ..., X_n \stackrel{iid}{\sim} \Gamma(a, \beta) = \frac{1}{\beta^a \Gamma(a)} x^{a-1} e^{-\frac{x}{\beta}}$ with a known. Show the best (most powerful) test of

$$H_0 : \beta = 1 \quad \text{vs.} \quad H_1 : \beta = 2$$

is a function of a sufficient statistic for β. Hence, find a level α most powerful test for H_0 against H_1. Is the test unbiased?
By the NP Lemma we know the form of the MP test and thus look at:

$$\begin{aligned} \frac{f(\vec{X} \mid \beta_1)}{f(\vec{X} \mid \beta_0)} &= \frac{\Gamma^{-n}(a)\beta_1^{-na} \prod x_i^{a-1} e^{-\beta_1^{-1} \sum x_i}}{\Gamma^{-n}(a)\beta_0^{-na} \prod x_i^{a-1} e^{-\beta_0^{-1} \sum x_i}} > k \\ &\Rightarrow e^{\left(\frac{1}{\beta_0} - \frac{1}{\beta_1}\right) \sum x_i} > k_2 \\ &\Rightarrow \sum x_i > k_3 \end{aligned}$$

Since $\sum x_i$ is sufficient for β when a is known (previous homework, etc), we've shown the MP test is a function of a sufficient statistic. Now to find the value of k_3 for a level-α test, note that since each X_i is a gamma, the sum of X's will also be gamma, and specifically $\Gamma(na, \beta)$ (from Stat 512). Hence we have that:

$$\begin{aligned} \alpha &= Pr[\sum x_i > k_3] \\ \Rightarrow 1 - \alpha &= Pr[\sum x_i \leq k_3] \\ \Rightarrow k_3 &= \text{the } 1 - \alpha^{th} \text{ quantile from a } \Gamma(na, \beta) \text{ distribution} \end{aligned}$$

Recall that for a test to be unbiased, then $E_{\theta_0 \in H_0}(\phi(\vec{X})) \leq E_{\theta_1 \in H_1}(\phi(\vec{X}))$.
This can be easily shown comparing $Pr[\Gamma(na, \beta_0) > k_3]$ vs. $Pr[\Gamma(na, \beta_1) > k_3]$.

(b) Repeat as in part (a), but for $H_0 : \beta = \beta_0$ versus $H_1 : \beta = \beta_1,\ \beta_1 > \beta_0$, and hence generlize the result from part (a) for a composite alternative. Also, show that the power curve is non-decreasing in β.
Since part (a) was solved without specifcally using the fact that $\beta_0 = 1$ and $\beta_1 = 2$, part (b) has already been solved (provided $\beta_0 < \beta_1$).
For showing the power being a non-decreasing function of β:

$$\begin{aligned} Pwr(\beta) &= P[\sum x_i > k_3 | \beta] \\ &= 1 - P[\Gamma(na, \beta) \leq k_3 | \beta] \\ &= 1 - P[\Gamma(na, 1) \leq \frac{k_3}{\beta} | \beta] \end{aligned}$$

Now this can be seen as being increasing in β.

- 9.3.2
Let $X_1, ..., X_n \overset{iid}{\sim} f(x|\theta) = e^{-(x-\theta)}1_{(\theta,\infty)}(x)$, the "unit displaced exponential". Use N-P Lemma to determine a level α most powerful (MP-α) test for $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1,\ \theta_1 > \theta_0$ Assume all $x_i > \theta_0$ to avoid dividing by zero, and since we shouldn't really accept either hypothesis if this happens.

$$\begin{aligned}
\frac{\exp(\sum -(x_i - \theta_1))1_{(\theta_1,\infty)}(min(x_i))}{\exp(\sum -(x_i - \theta_0))1_{(\theta_0,\infty)}(min(x_i))} &> c_1 \\
\exp(\sum -(x_i - \theta_1) + (x_i - \theta_0))1_{(\theta_1,\infty)}(min(x_i)) &> c_2 \\
T(\mathbf{x}) = 1_{(\theta_1,\infty)}(min(x_i)) &> c_3
\end{aligned}$$

Now c_3 can be anywhere in $(0, 1)$, because $T(\mathbf{x})$ only takes values $0, 1$. And the only α-level test we can derive is $\alpha = P_{H_0}(min(x_i) > \theta_1) = (P_{H_0}(X > \theta_1))^n = (1 - \exp(\theta_1 - \theta_0))^n$. This test, however, has power 1, since under the alternate hypothesis $X_i > \theta_1$.

- 9.3.5
Use the NP Lemma and show there is no Uniformly Most Powerful (UMP) α-level test for $H_0 : \sigma^2 = \sigma_0^2$ versus $H_1 : \sigma^2 \neq \sigma_0^2$ based on a random sample $X_1, ..., X_n \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$ with μ known. Also, show that the power function of the NP test for $H_0 : \sigma^2 = \sigma_0^2$ versus $H_1 : \sigma^2 = \sigma_1^2,\ \sigma_1^2 > \sigma_0^2$ changes as a function of $c = \sigma_1^2/\sigma_0^2$.

$$\begin{aligned}
\frac{f(\overrightarrow{X}\,|\sigma_1^2)}{f(\overrightarrow{X}\,|\sigma_0^2)} &= \left(\frac{\sigma_1^2}{\sigma_0^2}\right)^{\frac{-n}{2}} exp\left\{\frac{1}{2}\left(\frac{\sigma_1^2 - \sigma_0^2}{\sigma_0^2\sigma_1^2}\right)\sum(x_i - \mu)^2\right\} > k \\
&\Rightarrow\ (\sigma_1^2 - \sigma_0^2)\sum(x_i - \mu)^2 > k_2
\end{aligned}$$

It is clear at this point that the MP test will be through the test statistic $\sum(x_i - \mu)^2$, but which way does the inequality face? At this point we are unsure which of two situations we could be in, $\sigma_1^2 > \sigma_0^2$ or $\sigma_1^2 < \sigma_0^2$. Since this depends on the value of σ_1^2, we cannot have a UMP test.
In the second part of this problem, we'll take $\sigma_1^2 > \sigma_0^2$ as given. Thus, our test will reject for $T.S. = \sum(x_i - \mu)^2 > k_3$. And since each $X_i \sim \mathcal{N}(\mu, \sigma^2)$, then $\frac{T.S.}{\sigma^2} \sim \chi_n^2$ Thus, k_3 under the null hypothesis will be $\sigma_0^2 \cdot \chi^2_{n,(1-\alpha)}$. This in turn implies that under the alternative (to calculate the power),

$$Pr\left[\sum(x_i - \mu)^2 > k_3|\sigma_1^2\right] = Pr\left[\frac{\sum(x_i - \mu)^2}{\sigma_1^2} > \frac{k_3}{\sigma_1^2}|\sigma_1^2\right]$$

$$= Pr\left[\frac{\sum(x_i-\mu)^2}{\sigma_1^2} > \frac{\sigma_0^2\cdot\chi^2_{n,(1-\alpha)}}{\sigma_1^2}\right]$$

Now recognize the left side of the inequality above as another Chi-Squared distribution (remember, this is under the alternative, so dividing by σ_1^2 is appropriate here, whereas under the null, it was σ_0^2) and we can see the power being a function of the ratio σ_1^2/σ_0^2.

- 9.3.6

  Consider testing $H_0 : p(x) = p_0(x)$ versus $H_1 : p(x) = p_1(x)$ with a sample of size 1 using the following data:

x	2	4	6	8	10	12	14	16	18	20
$p_0(x)$	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.22	0.25	0.25
$p_1(x)$	0.04	0.04	0.09	0.16	0.20	0.24	0.08	0.05	0.05	0.05

  Find all critical regions of size $\alpha = 0.7$, and of these, determine which one has maximum power. Critical regions of size α (as opposed to level α) are: {2,4,8}, {2,12}, {4,10}, {6,8}, and {14} with respective power being 0.24, 0.28, 0.24, 0.25, and 0.08. Thus, the MP size α test here corresponds to a critical region of {2,12}. Note: the distinction of size versus level is just an equality versus inequality respectively. A size α test has type I error equal to α whereas a level α test has type I error $\le \alpha$.

- 9.3.8

  (a) For each of the following, find a UMP-α test (if one exists). Assume we have $X_1, ..., X_n$ in each case.

  (i) $H_0 : \theta \le \theta_0$ versus $H_1 : \theta > \theta_0,\ f(x|\theta) = \theta x^{-(\theta-1)} 1_{[1,\infty)}(x)$
  We'll start with NP-Lemma on $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1,\ \theta_1 > \theta_0$.

$$\frac{f(\vec{X}\,|\theta_1)}{f(\vec{X}\,|\theta_0)} = \frac{\theta_1^n\left(\prod x_i\right)^{-(\theta_1+1)}}{\theta_0^n\left(\prod x_i\right)^{-(\theta_0+1)}} > k$$
$$\Rightarrow \sum log(x_i) < k_2$$

  Now we need to figure out the distribution of $\sum log(X_i)$, which can be done with a simple change of variable. Let $Y = log(X_i)$. Then

$$
\begin{aligned}
f_Y(y|\theta) &= f_X(e^y|\theta)\cdot e^y = \theta e^{-y(\theta-1)}e^y 1_{(0,\infty)}(y) = \theta e^{-\theta y}1_{(0,\infty)}(y) = \mathcal{E}(\theta)\\
&\Rightarrow \sum log(x_i) \sim \Gamma(n, 1/\theta)\\
&\Rightarrow Pr\left[\sum log(x_i) < k_2|\theta_0\right] = \alpha\\
&\Rightarrow k_2 = \text{the } \alpha^{th} \text{ quantile from a } \Gamma(n, 1/\theta_0) \text{ distribution}
\end{aligned}
$$

Again, we actually have a UMP-α test over the alternative since we have yet to use θ_1 in derivations. Now if we just get the power function increasing in θ then we'll be done. (i.e. $Pwr(\theta) < Pwr(\theta_0)\ \forall\ \theta < \theta_0$)
This is true since we can write each as follows:

$$
\begin{aligned}
Pwr(\theta) &= Pr\left[\sum log(x_i) < k_2|\theta\right] = Pr\left[\Gamma(n,1) < \theta k_2\right]\\
Pwr(\theta_0) &= Pr\left[\sum log(x_i) < k_2|\theta_0\right] = Pr\left[\Gamma(n,1) < \theta_0 k_2\right]\\
&\Rightarrow Pwr(\theta) < Pwr(\theta_0) \text{ since } \theta < \theta_0
\end{aligned}
$$

Hence we have a UMP-α test for $H_0 : \theta \le \theta_0$ versus $H_1 : \theta > \theta_0$.

(ii) $H_0 : \theta \ge \theta_0$ versus $H_1 : \theta < \theta_0$, $f(x|\theta) = \frac{1}{\theta}x^{(1-\theta)/\theta}1_{(0,1)}(x)$
We'll start this one as in the previous part, with NP Lemma.

$$
\begin{aligned}
\frac{f(\vec{X}|\theta_1)}{f(\vec{X}|\theta_0)} &= \frac{\frac{1}{\theta_1}^n\left(\prod x_i\right)^{\frac{1-\theta_1}{\theta_1}}}{\frac{1}{\theta_0}^n\left(\prod x_i\right)^{\frac{1-\theta_0}{\theta_0}}} > k\\
&\Rightarrow \sum log(x_i) > k_2
\end{aligned}
$$

Note: Different direction of inequality in NP Lemma result compared to that in the previous part, this is because $\theta_1 < \theta_0$ this time.
Now we need to figure out the distribution of $\sum log(X_i)$ again, which can be done with a simple change of variable. Again let $Y = log(X_i)$. Then

$$
\begin{aligned}
f_Y(y|\theta) &= f_X(e^y|\theta)\cdot e^y = \frac{1}{\theta}e^{y\frac{1-\theta}{\theta}}e^y 1_{(-\infty,0)}(y) = \frac{1}{\theta}e^{-y\frac{1}{\theta}}1_{(0,\infty)}(y) = \mathcal{E}(1/\theta)\\
&\Rightarrow -\sum log(x_i) \sim \Gamma(n, \theta)\\
&\Rightarrow Pr\left[-\sum log(x_i) < k_2|\theta_0\right] = \alpha\\
&\Rightarrow k_2 = \text{the } \alpha^{th} \text{ quantile from a } \Gamma(n, \theta_0) \text{ distribution}
\end{aligned}
$$

Again, we actually have a UMP-α test over the alternative since we have yet to use θ_1 in derivations.
Now if we just get the power function decreasing in θ this time, then we'll be done. (i.e. $Pwr(\theta) < Pwr(\theta_0)\ \forall\ \theta > \theta_0$)

$$\begin{aligned}
Pwr(\theta) &= Pr\left[-\sum log(x_i) < k_2|\theta\right] = Pr\left[\Gamma(n,1) < k_2/\theta\right] \\
Pwr(\theta_0) &= Pr\left[-\sum log(x_i) < k_2|\theta_0\right] = Pr\left[\Gamma(n,1) < k_2/\theta_0\right] \\
&\Rightarrow Pwr(\theta) < Pwr(\theta_0) \text{ since } \theta > \theta_0
\end{aligned}$$

Hence we have a UMP-α test for $H_0 : \theta \geq \theta_0$ versus $H_1 : \theta < \theta_0$.

(b) Find expressions for the power functions in part (a).
These were specified above and denoted $Pwr(\theta)$.

- 9.3.11
Based on a random sample of size n, test $H_0 : f(x) = \phi(x)$ standard normal versus $H_1 : f(x) = \pi^{-1}(1+x^2)^{-1}1_{(-\infty,\infty)}(x)$ (Cauchy) using the NP Lemma to find a MP-α test. For $\alpha = 0.05$, find the critical region and power of the test you find.
Finding a critical region and power in closed form proved intractable in this problem, so I won't persist. Proceed to find the UMP test using NP lemma.

$$\begin{aligned}
\frac{(\pi)^{-n}\prod_{i=1}^{n}(1+x_i^2)^{-1}}{(2\pi)^{-n/2}\exp(-\sum_{i=1}^{n}x_i^2/2)} &> c \\
\exp(\sum_{i=1}^{n} -\log(1+x_i^2) + x_i^2/2) &> c_2 \\
T(\mathbf{x}) = \sum_{i=1}^{n} x_i^2/2 - \log(1+x_i^2) &> c_3
\end{aligned}$$

We can do the third step because exp is a strictly increasing function. Set c_3 such that $P_{H_0}(T(\mathbf{x}) > c_3) = \alpha$. Our critical region is the set of $(\mathbf{x} : T(\mathbf{x}) > c_3)$, and the power of this test is $P_{H_1}(T(\mathbf{x}) > c_3)$.

- 9.3.13

(a) Want to test $H_0 : \lambda = \lambda_0$ versus $H_1 : \lambda = \lambda_1,\ (\lambda_0 < \lambda_1)$ for a Poisson population. How large of a sample size is needed to obtain a MP-α test with power of $1-\beta$? (Assume sample size is CLT-worthy large)

We assume we can approximate this as $\overline{X} \sim$N$(\lambda, \lambda/n)$. By N-P,

$$\begin{aligned}
\frac{\prod \frac{\lambda_1^{x_i} e^{-\lambda_1}}{x_i!}}{\prod \frac{\lambda_0^{x_i} e^{-\lambda_0}}{x_i!}} &> c_1 \\
\left(\frac{\lambda_1}{\lambda_0}\right)^{\sum x_i} &> c_2 \\
\sum x_i(\log(\lambda_1) - \log(\lambda_0)) &> c_3 \\
T(\mathbf{x}) = \overline{x} &> c_4 \\
P_{H_0}(\overline{x} > c_4) &= P_{H_0}\left(\frac{(\overline{x} - \lambda_0)}{(\lambda_0/n)^{1/2}} > \frac{c_4 - \lambda_0}{(\lambda_0/n)^{1/2}}\right) \\
&\approx 1 - \Phi\left(\frac{c_4 - \lambda_0}{(\lambda_0/n)^{1/2}}\right) \\
&= \alpha \\
c_4 &= \Phi^{-1}(1-\alpha)(\lambda_0/n)^{1/2} + \lambda_0 \\
P_{H_1}(\overline{x} > c_4) &\approx 1 - \Phi\left(\frac{c_4 - \lambda_1}{(\lambda_1/n)^{1/2}}\right) \\
&= 1 - \beta \\
\Phi^{-1}(\beta) &= \frac{c_4 - \lambda_1}{(\lambda_1/n)^1/2} \\
\Phi^{-1}(\beta) &= \Phi^{-1}(1-\alpha)(\lambda_0/\lambda_1)^{1/2} + (\lambda_0 - \lambda_1)/(\lambda_1/n)^{1/2} \\
n &= \left(\frac{\lambda_1^{1/2}\Phi^{-1}(\beta) - \lambda_0^{1/2}\Phi^{-1}(1-\alpha)}{\lambda_0 - \lambda_1}\right)^2
\end{aligned}$$

(b) Calculate the sample size in part (a) if $\lambda_0 = 2, \lambda_1 = 1, \alpha = 0.05, 1 - \beta = 0.90$.
Note: there's a likely misprint here, we found a test for $\lambda_1 > \lambda_0$ and here $\lambda_1 = 1 < \lambda_0 = 2$.
Just going with it: n=13, switching it: n=12.