Stat 513
HW 5 Solutions
February 26, 2004

- 8.3.3
  Let $X_1, ..., X_n \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$.

  (a) Find a UMVUE for μ if σ^2 is known, and show that this estimator is CAN.
  Since σ^2 is known, we have $\sum x_i$ being a complete sufficient statistic for μ (as seen in class examples, previous homework or by writing out the likelihood in exponential family form). Thus, taking $\overline{X}$ as our estimator of μ, we have one that is unbiased, and based on a complete sufficient statistic and thus, by Theorem 8.3.8 (Lehman-Scheffe), is UMVUE for μ.
  Now to show it is CAN (also done before), we can just appeal to the Lindeberg-Lévy CLT and know that:
  $$\sqrt{n}(\overline{X} - \mu) \underset{d}{\rightarrow} \mathcal{N}(0, \sigma^2)$$

  (b) Find a UMVUE for σ^2 if μ is known, and show that this estimator is CAN.
  When μ is known, we can write the likelihood as follows:
  $$f(\overrightarrow{x}\ |\sigma^2) = exp\{-\frac{n}{2}ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum(x_i - \mu)^2\}$$
  Thus, $\sum(x_i - \mu)^2$ is complete and sufficient for σ^2 since we have the additional requirement of $\Theta = (0, \infty)$ containing a 1-dim rectangle.
  Now noting that $E(\sum(x_i - \mu)^2) = n\sigma^2$, we now know by the Lehman-Scheffe Theorem again that $T(\overrightarrow{X}) = \frac{1}{n}\sum(x_i - \mu)^2$ is UMVUE for σ^2 (when μ is known).
  Now for the CAN part, again by the Lindeberg-Levy CLT (and previous homework) we know that
  $$\sqrt{n}(\hat{\sigma^2} - \sigma^2) \underset{d}{\rightarrow} \mathcal{N}(0, 2\sigma^4)$$

- 8.3.6
  Let $X_1, ..., X_n \overset{iid}{\sim} \Gamma(\alpha - 1, \beta, 0)$.

  (a) Find a BUE for $\alpha\beta$.
  A good place to start is finding a comnplete sufficient statistic which is usually easy to find via exponential family form. In the context of this problem:
  $$f(\overrightarrow{x}\ |\alpha, \beta) = exp\{-n\ ln(\beta^\alpha\Gamma(\alpha)) + (\alpha - 1)ln\left(\prod x_i\right) - \frac{1}{\beta}\sum x_i\}$$
  From here we can see that $(\prod x_i, \sum x_i)$ is jointly complete and sufficient for (α, β) since we have the proper exponential family form and $\Theta = (0, \infty)$ x $(0, \infty)$ contains a two-dimensional

rectangle. Conveniently, $E(X_i) = \alpha\beta$ (using the book's parameterization), which means that $\overline{X}$ is unbiased for $\alpha\beta$, and this is a function of our joint-complete sufficient statistics (specifically $0 \cdot \prod x_i + \frac{1}{n} \cdot \sum x_i$. Now appealing to the Theorem by Lehman-Scheffe again, we have $T(\vec{X}) = \overline{X}$ is UMVUE (hence BUE) for $\alpha\beta$.

(b) Show estimator from part (a) is CAN.
Lindebeger-Levy CLT here too:

$$\sqrt{n}(\overline{X} - \alpha\beta) \underset{d}{\rightarrow} \mathcal{N}(0, \alpha\beta^2)$$

- 8.3.7
Let $X_1, ..., X_n \overset{iid}{\sim} \mathcal{E}(\theta) = \theta e^{-\theta x} 1_{[0,\infty)}(x)$. Show $(n-1)/\sum X_i$ is the unique MVUE.

$$\begin{aligned} f_X(x) &= \prod_{i=1}^{n} \theta e^{-\theta x_i} 1_{[0,\infty)}(x_i) \\ &= e^{(\theta \sum x_i - n \log \theta)} 1_{[0,\infty)}(min(x_i)) \end{aligned}$$

This is exponential family form, with $h(x) = 1_{[0,\infty)}(min(x_i))$, $T(X) = \sum x_i$, $\tau(\theta) = \theta$ and $A(\theta) = n \log \theta$. Θ=(the set of all possible θs) = $(0, \infty)$. This contains a 1-d open rectangle (e.g. (0,1)). Therefore we can apply our theorem about exponential families and complete sufficient statistics. $\sum x_i$ is the complete sufficient statistic for θ. Now $(n-1)/\sum X_i$ is a function of this complete sufficient statistic, so if it has expectation θ, it must be the UMVUE. Recall that the sum of n exponential(θ)s is Gamma(n,θ). Let $Y = \sum X_i \sim Gamma(n, \theta)$

$$\begin{aligned} E\left(\frac{n-1}{Y}\right) &= \int_0^\infty \frac{n-1}{y} \cdot \frac{y^{(n-1)}\theta^n}{\Gamma(n)} e^{-\theta y} dy \\ &= \theta \int_0^\infty \frac{y^{(n-2)}\theta^{(n-1)}}{\Gamma(n-1)} e^{-\theta y} dy \\ &= \theta \end{aligned}$$

Since the form in the integral is the density of a Gamma(n-1,θ). (Here I used the rate paramaterization of a Gamma).

- 8.3.9
In problem 8.2.15, find a BUE of $\mu/(1 - e^{-\mu})$ and $1 - e^{-\mu}$.

From 8.2.15: Let $X_1, ..., X_n \overset{iid}{\sim}$ truncated Poisson where $p(x|\mu) = e^{-\mu}\mu^x/(x!(1-e^{-\mu}))1_{\{1,2,...\}}(x)$. Also from 8.2.15, we know that $\sum x_i$ is complete and sufficient for μ under this distribution. Now noting that $E(X) = \mu/(1-e^{-\mu})$, it follows immediately from Lehman-Scheffe that $\overline{X}$ is UMVUE for $\mu/(1-e^{-\mu})$.
Now for $(1-e^{-\mu})$. First find an unbiased estimator based on only one observation.

$$\begin{aligned}
E(t(X_1)) &= (1-e^{-\mu}) \\
\sum_{k=1}^{\infty} \frac{t(k)\mu^k e^{-\mu}}{k!(1-e^{-\mu})} &= (1-e^{-\mu}) \\
\sum_{k=1}^{\infty} \frac{t(k)\mu^k}{k!} &= e^{\mu} - 2 + e^{-}\mu \\
\sum_{k=1}^{\infty} \frac{t(k)\mu^k}{k!} &= \sum_{1}^{\infty} \frac{\mu^k}{k!} + \sum_{1}^{\infty} \frac{(-\mu)^k}{k!} \\
\sum_{k=1}^{\infty} \frac{t(k)\mu^k}{k!} &= \sum_{1}^{\infty} (1+(-1)^k)\frac{\mu^k}{k!} \\
t(X_1) = 1 + (-1)^{X_1}
\end{aligned}$$

Use the refinement theorem to obtain the BUE of $1-e^{-\mu}$: $T(X) = E(T(X_1)|\sum X_i)$. The sum of X_i do not have a particularly nice distribution, so we leave it at this.

- 8.3.11
  Let X be a single observation from a Poisson distribution with unknown parameter μ.

  (a) Find a BUE for μ^2. [Hint: Note that $E(X(X-1)) = \mu^2$, work it out using $Var(X) = E(X^2) - (E(X))^2$ if you don't see it immediately]
  First note that when there's just a single observation, the complete sufficient statistic for μ in a Poisson distribution is just x itself. Showing this via exponential family form:

  $$f(\vec{x}\,|\mu) = f(x|\mu) = exp\{x\ ln(\mu) - \mu - ln(x!)\}$$

  and we note $\Theta = (0, \infty)$ contains the necessary 1-dim rectangle. Now, in concordance with the hint, we know $T(\vec{X}) = T(X) = x(x-1)$ is UMVUE (via Lehman-Scheffe) for μ^2

  (b) If we have a random sample of size n, find the BUE for μ^2.
  For a sample of size n, our complete sufficient statistic for μ is $S_n = \sum x_i$, and we know the distribution of this statistic will follow $\mathcal{P}(n\mu)$. Now in similar form to part (a), we can see $E(S_n(S_n-1)) = n^2\mu^2$ so if we take $T(\vec{X}) = \frac{1}{n^2}S_n(S_n-1)$ then we have an unbiased estimator of μ^2 based on the complete sufficient statistic for μ and will thus be UMVUE (BUE) by Lehman-Scheffe.

(c) Based on a sample of size 1, find the BUE for $\mu^r, r > 1$. [Hint: Compute $E(X(X-1)(X-2)...(X-r+1))$]
Back to the situation as in part (a) with just x being complete and sufficient. Now, as suggested by the hint, if we calculate the expceted value there (very similar to hmwrk 3 in 512) we'll see that expression is unbiased for μ^r and we have the BUE since it's a function of our complete sufficient statistic.

- 9.2.1
An urn has 10 balls with θ of them blue (the rest being whatever color you want, other than blue). We're in the following hypothesis testing situation:

$$H_0 : \theta = 3 \quad \text{vs.} \quad H_1 : \theta = 4$$

Suppose a sample consists of 3 balls, and the decision rule is to reject H_0 if all 3 balls in the sample are blue. Compute the errors α (Type I) and β (Type II) when:

(a) Sampling is done without replacement.

$$\begin{aligned} \alpha &= P(\text{ all three balls are blue} \mid \theta = 3) \\ &= \frac{\theta}{10} \cdot \frac{\theta-1}{9} \cdot \frac{\theta-2}{8} = \frac{\theta(\theta-1)(\theta-2)}{720} = \frac{1}{120} \end{aligned}$$

$$\begin{aligned} \beta &= 1 - P(\text{ all three balls are blue} \mid \theta = 4) \\ &= 1 - \frac{1}{30} = \frac{29}{30} \end{aligned}$$

(b) Sampling is done with replacement.

$$\begin{aligned} \alpha &= P(\text{ all three balls are blue} \mid \theta = 3) \\ &= \left(\frac{\theta}{10}\right)^3 = \frac{27}{1000} \end{aligned}$$

$$\begin{aligned} \beta &= 1 - P(\text{ all three balls are blue} \mid \theta = 4) \\ &= 1 - \frac{64}{1000} = \frac{936}{1000} \end{aligned}$$

- 9.2.5
Suppose we have a random sample of size 5 from a Poisson distribution with mean $\lambda \in \{2, 3\}$. We want to test the following:

$$H_0 : \lambda = 3 \quad \text{vs.} \quad H_1 : \lambda = 2$$

with rejection when $\overline{X} < c$. Find the critical region to use for this test if α is set at 0.05 (or as close to 0.05 as possible using a Poisson Table).
The appropriate critical region will be such that

$$Pr(\overline{X} < c|\lambda = 3) = Pr(\sum_{i=1}^{5} x_i < 5c|\lambda = 3)$$

Now since we know that the sum of n independent poisson r.v.'s with common support will be distributed as poisson with $n\lambda$, then we know what the distribution of our test statistic $\sum x_i$ is. With respect to the poisson table on page 770, we can see that a value of 9 will keep the test at a level 0.05. This implies the value of c we're looking for is $9/5 = 1.8$.