Stat 513
HW 4 Solutions
February 25, 2004

- 8.1.4
Let $X_1, ..., X_n \overset{iid}{\sim} \Gamma(\alpha, \beta) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}}$. Find the minimal sufficient statistics for:
First note:

$$\frac{\Gamma(\overrightarrow{X} \,|\alpha, \beta)}{\Gamma(\overrightarrow{Y} \,|\alpha, \beta)} = \left(\frac{\prod x_i}{\prod y_i}\right)^\alpha e^{-\frac{\sum x_i - \sum y_i}{\beta}}$$

  (a) and (b)
  Clearly by the expression above, we have minimal sufficient statistics of $\prod x_i$ for $\alpha$ when $\beta$ is known, and $\sum x_i$ for $\beta$ when $\alpha$ is known.

  (c) Show that MLE's in parts (a) and (b) are functions of the statistics in each part.
  MLE for $\alpha$ when $\beta$ is known is the solution to

$$-n \; ln(\beta) - digamma(\hat{\alpha}) + \sum ln(x_i) = 0$$

  this is clearly a function of our sufficient statistic from part (a) since $\sum ln(x_i) = ln(\prod x_i)$

  MLE for $\beta$ when $\alpha$ is known is

$$\frac{\overline{X}}{\alpha}$$

  and this is clearly a function of the minimal sufficient statistic found in part (b).

Note: The method shown above is not unique for solving this problem. The Gamma distribution is a member of the exponential family (class) and thus we could have used this fact in solving for the sufficient statistics, and then could immediately have a confirmation on minimum sufficiency and completeness. However, the method above is not useless, not all distributions are exponential family, and then those results do not apply.

- 8.1.11
Let $X_1, ..., X_n \overset{iid}{\sim} LN(\mu, \sigma^2)$. Find the joint sufficient statistics for $(\mu, \sigma^2)$, and show that the MLE's are functions of this joint statistic.
We know already that for a r.v. $Y = ln(X) \sim \mathcal{N}(\mu, \sigma^2)$, the sufficent statistic for $\theta = (\mu, \sigma^2)$ is $T(\overrightarrow{Y}) = (\sum y_i, \sum y_i^2)$. Thus our sufficient statistic for this problem will be $T(\overrightarrow{X}) = (\sum ln(x_i), \sum(ln(x_i))^2)$.

$$\begin{aligned}\hat{\mu} &= \frac{\sum ln(x_i)}{n} \quad \text{(fxn of sufficient statistic above)} \\ \hat{\sigma^2} &= \frac{\sum ln(x_i) - \hat{\mu})^2}{n} \quad \text{(also a fxn of sufficient statistic above)}\end{aligned}$$

- 8.1.12
  Let $X_1, ..., X_n \overset{iid}{\sim} f(x|\theta) = \theta(1+x)^{-(1+\theta)} 1_{(0,\infty)}(x)$ for $\theta > 0$. Show that $\prod_{i=1}^n (1+X_i)$ is a minimal sufficient statistc for $\theta$, and that the MLE is a function of this statistic.

  $$\frac{\prod_{i=1}^n \theta(1+x)^{-(1+\theta)} 1_{(0,\infty)}(x)}{\prod_{i=1}^n \theta(1+y)^{-(1+\theta)} 1_{(0,\infty)}(y)} = \frac{\theta^n \left(\prod_{i=1}^n (1+x_i)\right)^{-(1+\theta)} 1_{(0,\infty)}(x_i)}{\theta^n \left(\prod_{i=1}^n (1+y_i)\right)^{-(1+\theta)} 1_{(0,\infty)}(y_i)}$$

  Now we can see that first, $\prod_{i=1}^n (1+x_i)$ is a sufficient by the 'factorization theorem', and second, that it is mininmal since the expression will be independent of $\theta$ iff $\prod_{i=1}^n (1+x_i) = \prod_{i=1}^n (1+y_i)$.

  Solving for the MLE for $\theta$,

  $$\hat{\theta} = \frac{n}{\sum_{i=1}^n ln(1+x_i)} = \frac{n}{ln(\prod_{i=1}^n (1+x_i))}$$

  and we can see this is a function of the minimal sufficient statistic found above. Note: "n" is in the numerator of our MLE, but that's okay! Sometimes it happens, as shown here.

- 8.2.4

  (a) Let $X_1, X_2 \overset{iid}{\sim} f(x|\theta) = \frac{1}{\theta} e^{\frac{-x}{\theta}} 1_{(0,\infty)}(x)$. Show that $T_1(X_1, X_2) = X_1 + X_2$, and $T_2(X_1, X_2) = \frac{X_1}{X_1+X_2}$ are independent r.v.'s. [Hint: Use Basu's Theorem (8.2.23)].
  First thing to do is identify what distribution we're working with here, it's an exponential with mean paramter $\theta$. To use Basu's theorem, we'll need to show two things: that one of $T_1$, and $T_2$ is complete and sufficient, and the other is ancillary (not just first-order ancillary).
  We already know $\sum_{i=1}^n x_i$ is sufficient for $\theta$ when we have $X_1, ..., X_n \overset{iid}{\sim} \mathcal{E}(\theta)$ from class. Thus, $T_1$ is sufficient here since we have $n = 2$. Now we just show $T_2$ is ancillary, that is, the distribution of $T_2$ is independent of $\theta$.
  First consider $Y_i = X_i/\theta$. A simple variable transformtion shows $Y_i \sim \mathcal{E}(1)$ which is clearly independent of $\theta$ (see notes from 512, it's in there). Now note

  $$T_2 = \frac{X_1}{X_1 + X_2} = \frac{\theta Y_1}{\theta(Y_1 + Y_2)} = \frac{Y_1}{Y_1 + Y_2}$$

Since the distribution of $Y_1/(Y_1+Y_2)$ is independent of $\theta$, and this equals $T_2$, then the distribution of $T_2$ must also be indpendent of $\theta$, and thus ancillary. Now by Basu's theorem, we have that $T_1$, and $T_2$ are independent.

Slight aside: The exponential distribution (non-shifted one) belongs to a family called a "scale family". This is because they're all related by a scaled factor (as we saw above, by $\theta$). There are also location families, and location-scale families. Similar to the method above for showing ancillarity, for location families we'll look for a difference between two r.v.'s, and for location-scale families we'll have both a difference and ratio (ex. $\frac{X_1-X_2}{\sum X_i - X_2}$).

(b) Generalize the result in part (a) by proving:
If $X_1, ..., X_n \overset{iid}{\sim} f(part(a))$, then $S_n = X_1 + ... + X_n$ and $T_i = \frac{X_i}{S_n}$ are independent $\forall\, i \in \{1, ..., n\}$.
Well we already said above that $S_n = \sum x_i$ is sufficient. Now we just note that instead of $Y_1$, and $Y_2$ we now have $Y_1, ..., Y_n$ and we're done using the same argument(s) as before.

- 8.2.5
Let $X_1, ..., X_n \overset{iid}{\sim} \mathcal{U}(0,\theta)$ and $Y_i$ denote the $i^{th}$ order statistic of the sample. Show that the distribution of $T_n = \sum_{i=1}^n X_i/Y_n$ does not depend on $\theta$, hence that $T_n$ is independent of $Y_n$.
One way to do this, as suggested by the hint, is through induction. But it kind of sucks, and there's an easier way.
First you have to note that the distribution we have here is a scale family as in the previous problem. Let $Z_i = \frac{X_i}{\theta}$

$$\begin{aligned}
\frac{dx}{dz} &= \theta \\
f_Z(z) &= f_X(\theta \cdot z) \cdot \theta \\
&= \frac{\theta}{\theta} 1_{(0 \le \theta \cdot z \le \theta)} \\
&= 1_{(0 \le z \le 1)} \\
&= \mathcal{U}(0,1) \\
T_n &= \sum_{i=1}^n \frac{X_i}{X_{(n)}} \\
&= \sum_{i=1}^n \frac{\theta Z_i}{\theta Z_{(n)}} \\
&= \sum_{i=1}^n \frac{Z_i}{Z_{(n)}}
\end{aligned}$$

Since $Z_i$ are independent of $\theta$, and $T_n$ can be expressed as only a function of the $Z_i$'s without $\theta$, $T_n$ must be an ancillary statistic. An example in the book showed that $X_{(n)}$ is a complete sufficient statistic for $\theta$, so by Basu's theorem, $Y_n = X_{(n)}$ and $T_n$ are independent.

- 8.2.9
  Let $X_1, ..., X_n$ be as in problem 8.1.8, that is, $\overset{iid}{\sim}$ discrete $\mathcal{U}(1, \theta)$. $Y_n = max(X_1, ..., X_n)$ is the minimal suffcient statistic for $\theta$. Show $Y_n$ is also complete.

$$\begin{aligned}
P(Y_n \le k) &= P(X \le k)^n \\
&= \frac{k^n}{\theta} \\
P(Y_n = k) &= P(Y_n \le k) - P(Y_n \le k-1) \\
&= \theta^{-n}(k^n - (k-1)^n) \\
E(u(Y_n)) &= \sum_{k=1}^{\theta} u(k)\theta^{-n}(k^n - (k-1)^n) \\
&= \theta^{-n} \sum_{k=1}^{\theta} u(k)(k^n - (k-1)^n) \\
E(u(Y_n)) = 0 &\iff \sum_{k=1}^{\theta} u(k)(k^n - (k-1)^n) = 0
\end{aligned}$$

We show $u(k)$ is identically zero by induction.

$$\begin{aligned}
\text{Let } \theta &= 1 \\
u(1) \cdot 1 &= 0 \\
u(1) &= 0 \\
\text{Assume } u(k) &= 0 \text{ for } k \in (1, .., \theta - 1) \\
\sum_{k=1}^{\theta} u(k)(k^n - (k-1)^n) &= u(\theta)(theta^n - (theta - 1)^n) \\
= 0 &\iff u(\theta) = 0
\end{aligned}$$

Since $\theta^n - (\theta - 1)^n \neq 0$ for $n > 0$. Therefore, $Y_n$ is complete, since no function of it is an unbiased estimator of zero.

- 8.2.15
  Let $X_1, ..., X_n \overset{iid}{\sim}$ truncated Poisson where $p(x|\mu) = e^{-\mu}\mu^x/(x!(1 - e^{-\mu}))1_{\{1,2,...\}}(x)$.

  (a) Show the joint distribution of $X_1, .., X_n$ is a member of the exponential class.

$$
\begin{aligned}
f(\vec{X}\,|\mu) &= \frac{e^{-n\mu}\mu^{\sum x_i}}{\prod x_i!}(1-e^{-\mu})^{-n} = e^{ln\left(\frac{e^{-n\mu}\mu^{\sum x_i}}{\prod x_i!}(1-e^{-\mu})^{-n}\right)} \\
&= e^{\sum x_i ln(\mu) - ln(\prod x_i!) - n(\mu + ln(1-e^{\mu})}
\end{aligned}
$$

Now we can see that we have the exponential family form where using the notation from D&M: $\phi_0(\vec{X}) = -ln(\prod x_i!)$, $c_0(\mu) = -n(\mu + ln(1-e^{\mu})$, $\phi_1(\vec{X}) = \sum x_i$, $c_1(\mu) = ln(\mu)$.

(b) Find a complete minimal sufficient statistic for $\mu$.
From the form above and since our parameter space for $\mu$, $\Theta = (0, \infty)$ contains a one-dimensional rectangle (interval), we can see that $\sum x_i$ is not only a sufficient statistic, but also minimal and complete.

- 8.2.17
Let $X_1, ..., X_n \overset{iid}{\sim} \mathcal{N}(\mu, \sigma^2)$. Show $\bar{X}_n$ and $\sum_{i=n}^{1} a_i X_i$ are independent r.v.'s if $\sum_{i=n}^{1} a_i = 0$.
We already know that $\sum x_i$ is complete and minimally sufficient for $\mu$ when $\sigma^2$ is known (and hence $\bar{X}_n$). Now to use Basu's Theorem, we would just need to show that $\sum_{i=n}^{1} a_i X_i$ is ancillary with respect to $\mu$.
Note that since each $x_i$ is independently normally distributed, $\sum_{i=n}^{1} a_i X_i \sim \mathcal{N}(\sum_{i=n}^{1} a_i \mu, \sum_{i=n}^{1} a_i^2 \sigma^2) = \mathcal{N}(\mu \sum_{i=n}^{1} a_i, \sigma^2 \sum_{i=n}^{1} a_i^2)$. Now this will clearly be indpendent of $\mu$ (and thus ancillary) iff $\sum_{i=1}^{n} a_i = 0$. Thus by Basu's Theorem, $\bar{X}_n$ and $\sum_{i=n}^{1} a_i X_i$ are independent r.v.'s if $\sum_{i=n}^{1} a_i = 0$.