Stat 512
HW 7 Solutions
December 1, 2003

- 6.3.2
  Let $X_1, X_2, \ldots$ be iid r.v.'s $\sim f_X(x) = \frac{1}{\lambda} e^{\frac{-x}{\lambda}} 1_{[x>0]}$ with $0 < \lambda < \infty$

  (a) Show that

$$\frac{\sqrt{n}(\overline{X} - \lambda)}{\overline{X}} \xrightarrow[n\to\infty]{d} N(0,1)$$

$$\begin{aligned} E(X) &= \lambda \text{ and } Var(X) = \lambda^2 \\ &\Rightarrow E(\overline{X}) = \lambda \text{ and } Var(\overline{X}) = \frac{\lambda^2}{n} \\ &\Rightarrow \overline{X} \xrightarrow[n\to\infty]{p} \lambda \\ &\Rightarrow \frac{\lambda}{\overline{X}} \xrightarrow[n\to\infty]{p} \frac{\lambda}{\lambda} = 1 \end{aligned}$$

  Now we have

$$\frac{\sqrt{n}(\overline{X} - \lambda)}{\overline{X}} = \frac{\sqrt{n}(\overline{X} - \lambda)}{\lambda} \cdot \frac{\lambda}{\overline{X}} \xrightarrow[n\to\infty]{d} N(0,1) \cdot 1 = N(0,1)$$

  (b) Let $\beta = \frac{1}{\lambda}$. Show that

$$\sqrt{n}(\beta\overline{X} - 1) \xrightarrow[n\to\infty]{d} N(0,1)$$

$$\sqrt{n}(\beta\overline{X} - 1) = \frac{\sqrt{n}(\overline{X} - \frac{1}{\beta})}{\frac{1}{\beta}} = \frac{\sqrt{n}(\overline{X} - \lambda)}{\lambda} \xrightarrow[n\to\infty]{d} N(0,1)$$

- 6.3.3
  Let $X_1, X_2, \ldots$ be iid $N(1,1)$ r.v.'s. Prove or disprove:

$$\sqrt{n}(\overline{X}^2 - 1) \xrightarrow[n\to\infty]{d} N(0,1)$$

  There's more than one way to complete this problem.

(1) Slutsky

$$\begin{array}{rcl}
\sqrt{n}(\overline{X}^2 - 1) & = & \sqrt{n}(\overline{X} - 1)(\overline{X} + 1) \\
\sqrt{n}(\overline{X} - 1) & \xrightarrow[n\to\infty]{d} & N(0,1) \text{ by CLT} \\
(\overline{X} + 1) & \xrightarrow[p]{} & 2 \\
\Longrightarrow \sqrt{n}(\overline{X}^2 - 1) & \xrightarrow[n\to\infty]{d} & 2 \cdot N(0,1) \neq N(0,1)
\end{array}$$

(2) Delta Method
Use the continuous function $g(x) = x^2$ $(g'(x) = 2x)$.

$$\begin{array}{rcl}
\sqrt{n}(\overline{X} - 1) & \xrightarrow[n\to\infty]{d} & N(0,1) \text{ by CLT} \\
\Rightarrow \sqrt{n}(\overline{X}^2 - 1^2) & \xrightarrow[n\to\infty]{d} & 2(1)N(0,1) \neq N(0,1)
\end{array}$$

- 6.3.5
  Let $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ be sequences of r.v.'s such that $X_m \xrightarrow{d} X$ and $Y_m \xrightarrow{d} X$. Prove or disprove: $X_m - Y_m \xrightarrow{d} 0$

  This statement is incorrect, thus a counterexample will suffice.
  Let $X_m \sim N(\frac{1}{m}, 1) \xrightarrow{d} N(0,1)$ and $Y_m \sim N(\frac{2}{m}, 1) \xrightarrow{d} N(0,1)$.
  Then $X_m - Y_m \xrightarrow{d} N(0,2) \neq N(0,1)$.

- 6.3.6
  Find an approximation to the probability $P(175 \leq \chi^2_{200} \leq 225)$.
  Using the normal approximation as in example 6.3.7 on page 320:

$$\begin{array}{rl}
\approx & P(175 \leq N(200, 400) \leq 225) \\
= & P\left(\dfrac{(175-200)}{\sqrt{400}} \leq N(0,1) \leq \dfrac{(225-200)}{\sqrt{400}}\right) \\
= & \Phi\left(\dfrac{(225-200)}{\sqrt{400}}\right) - \Phi\left(\dfrac{(175-200)}{\sqrt{400}}\right) \\
= & \Phi(1.25) - \Phi(-1.25) \approx 0.7887
\end{array}$$

- 6.3.7
  Let $X_1, X_2, \ldots, X_{n_1}$ be a random sample $\sim N(\mu_1, \sigma^2)$ of size $n_1$ and $Y_1, Y_2, \ldots, Y_{n_2}$ be a random sample $\sim N(\mu_2, \sigma^2)$ of size $n_2$ with $\sigma > 0$. Then given the pooled variance estimate of

$$s_p^2 = \frac{\sum_{i=1}^{n_1}(X_i - \overline{X})^2 + \sum_{j=1}^{n_2}(Y_j - \overline{Y})^2}{n_1 + n_2 - 2}$$

prove the following as $\min(n_1, n_2) \longrightarrow \infty$:

(a) $\overline{X} - \overline{Y} \xrightarrow[p]{} \mu_1 - \mu_2$

$$\begin{aligned} E(X) &= \mu_1 \text{ and } E(Y) = \mu_2 \\ Var(X) &= \sigma^2 = Var(Y) = \sigma^2 \end{aligned}$$

Now the result follows by WLLN and Corollary 6.3.14(i).

(b) $s_p^2 \xrightarrow[p]{} \sigma^2$
Assume that $n_1 = \lambda n_2$. We know from previous homework that both $s_x^2$ and $s_y^2 \xrightarrow[p]{} \sigma^2$.

$$\begin{aligned} s_p^2 &= \frac{(n_1 - 1)s_x^2}{n_1 + n_2 - 2} + \frac{(n_2 - 1)s_y^2}{n_1 + n_2 - 2} \\ &= \frac{(\lambda n_2 - 1)s_x^2}{n_2(\lambda + 1) - 2} + \frac{(n_2 - 1)s_y^2}{n_2(\lambda + 1) - 2} \\ &\xrightarrow[p]{} \frac{\lambda}{(\lambda + 1)}\sigma^2 + \frac{1}{(\lambda + 1)}\sigma^2 = \sigma^2 \end{aligned}$$

(c) $\frac{(\overline{X} - \overline{Y}) - (\mu_1 - \mu_2)}{\sqrt{s_p^2(\frac{1}{n_1} + \frac{1}{n_2})}} \xrightarrow{d} N(0, 1)$.
Assume again that $n_1 = \lambda n_2$. Now combining results from parts (a) and (b):

$$\begin{aligned} \frac{(\overline{X} - \overline{Y}) - (\mu_1 - \mu_2)}{\sqrt{s_p^2(\frac{1}{n_1} + \frac{1}{n_2})}} &= \frac{\sqrt{\sigma^2}}{\sqrt{s_p^2}} \cdot \frac{(\overline{X} - \mu_1)}{\sqrt{\sigma^2}\sqrt{\frac{\lambda+1}{\lambda n_2}}} - \frac{\sqrt{\sigma^2}}{\sqrt{s_p^2}} \cdot \frac{(\overline{Y} - \mu_2)}{\sqrt{\sigma^2}\sqrt{\frac{\lambda+1}{\lambda n_2}}} \\ &\xrightarrow{d} 1 \cdot \frac{N(0, \sigma^2)}{\sqrt{\sigma^2(\lambda + 1)}} - 1 \cdot \frac{N(0, \sigma^2)}{\sqrt{\sigma^2(\frac{\lambda+1}{\lambda})}} \end{aligned}$$

Now since linear combinations of normal r.v.'s are normal, and we know the two we have here are independent, we just need to check the expectation and variance:

$$\begin{aligned}
\text{Expectation} &= 0 - 0 = 0 \\
\text{Variance} &= \frac{1}{\sigma^2(\lambda+1)}\sigma^2 + \frac{\lambda}{\sigma^2(\lambda+1)}\sigma^2 = 1
\end{aligned}$$

thus, we have $N(0,1)$.

- 6.3.11
  Let $X_1, X_2, ...$ and $Y_1, Y_2, ...$ iid Bernoulli r.v.'s with means $p_1$ and $p_2$ respectively and $\hat{p}_{1,n_1}, \hat{p}_{2,n_2}$ be estimates for the paramters $p_1$ and $p_2$ via sample means of size $n_1$ and $n_2$ respectively.
  Prove the following:

$$\frac{(\hat{p}_{1,n_1} - \hat{p}_{2,n_2}) - (p_1 - p_2)}{\sqrt{\frac{\hat{p}_{1,n_1}(1-\hat{p}_{1,n_1})}{n_1} + \frac{\hat{p}_{2,n_2}(1-\hat{p}_{2,n_2})}{n_2}}} \xrightarrow[n\to\infty]{d} N(0,1)$$

This will be done in the same format as the previous problem.
We will assume that $n_1 = \lambda n_2$. Then we have:

$$\begin{aligned}
\hat{p}_{1,n_1} - \hat{p}_{2,n_2} &\xrightarrow[p]{} p_1 - p_2 \text{ by WLLN} \\
\hat{p}_{1,n_1}(1-\hat{p}_{1,n_1}) + \hat{p}_{2,n_2}(1-\hat{p}_{2,n_2}) &\xrightarrow[p]{} p_1(1-p_1) + p_2(1-p_2) \text{ by Slutsky}
\end{aligned}$$

$$\begin{aligned}
\frac{\sqrt{p_1(1-p_1)+\lambda p_2(1-p_2)}}{\sqrt{\hat{p}_{1,n_1}(1-\hat{p}_{1,n_1})+\lambda\hat{p}_{2,n_2}(1-\hat{p}_{2,n_2})}} &\frac{\sqrt{\lambda n_2}(\hat{p}_{1,n_1}-p_1)+\sqrt{\lambda}\sqrt{n_2}(\hat{p}_{2,n_2}-p_2)}{\sqrt{p_1(1-p_1)+\lambda p_2(1-p_2)}} \\
&\xrightarrow[n\to\infty]{d} 1 \cdot \frac{N(0,Var(X_1))+\sqrt{\lambda}N(0,Var(Y_1))}{\sqrt{Var(X_1)+\lambda Var(Y_1)}}
\end{aligned}$$

As before, linear combinations of normal r.v.'s are normal, and we know the two we have here are independent, so we just need to check the expectation and variance again:

$$\begin{aligned}
\text{Expectation} &= 0 - 0 = 0 \\
\text{Variance} &= \frac{Var(X_1)+\lambda Var(Y_1)}{Var(X_1)+\lambda Var(Y_1)} = 1
\end{aligned}$$

thus, we have $N(0,1)$.

- 6.3.17
  Let $\overline{X}_{100}$ denote the average of 100 rolls of a six-sided die. Compute the approximate probability for the following (we'll chose our approximation appealing to the CLT and use a normal distribution).
  Note: $E(X) = 3.5$ and $Var(X) = \frac{91}{6} - \left(\frac{7}{2}\right)^2 \approx 2.91667$.

  (a) $P(\overline{X}_{100} \leq 3.5)$

$$\begin{aligned} &= P\left(\underbrace{\frac{10(\overline{X}_{100} - 3.5)}{\left(\frac{91}{6} - \left(\frac{7}{2}\right)^2\right)}}_{Z} \leq \frac{10(3.5 - 3.5)}{\left(\frac{91}{6} - \left(\frac{7}{2}\right)^2\right)}\right) \\ &= P(Z \leq 0) = \frac{1}{2} \end{aligned}$$

  (b) $P(2.3 \leq \overline{X}_{100} < 3.4)$

$$\begin{aligned} &= P\left(\frac{10(2.3 - 3.5)}{\left(\frac{91}{6} - \left(\frac{7}{2}\right)^2\right)} \leq \underbrace{\frac{10(\overline{X}_{100} - 3.5)}{\left(\frac{91}{6} - \left(\frac{7}{2}\right)^2\right)}}_{Z} < \frac{10(3.4 - 3.5)}{\left(\frac{91}{6} - \left(\frac{7}{2}\right)^2\right)}\right) \\ &\approx P(-7.026 \leq Z < -.58554) \approx 0.2791 \end{aligned}$$