Stat 512
HW 6 Solutions
November 16, 2003

- 6.2.1
  Let X_i, be iid r.v.'s with $\mu, \sigma^2 < \infty$,

  (a) Show $\frac{2}{n(n+1)} \sum_{i=1}^n iX_i \underset{p}{\rightarrow} \mu$

$$
\begin{aligned}
E\left(\frac{2}{n(n+1)}\sum_{i=1}^n iX_i\right) &= \frac{2}{n(n+1)}\sum_{i=1}^n i \cdot E(X_i) \\
&= \mu \cdot \frac{2}{n(n+1)}\sum_{i=1}^n i \\
&= \mu \\
Var\left(\frac{2}{n(n+1)}\sum_{i=1}^n iX_i\right) &= \frac{4}{n^2(n+1)^2}\sum_{i=1}^n i^2 \cdot Var(X_i) \\
&= \sigma^2 \frac{4}{n^2(n+1)^2}\sum_{i=1}^n i^2 \\
&= \sigma^2 \frac{2n(n+1)(2n+1)}{3n^2(n+1)^2} \\
&\underset{n\to\infty}{\longrightarrow} 0
\end{aligned}
$$

  So by Chebyshev,

$$
\begin{aligned}
P\left(\left|\frac{2}{n(n+1)}\sum_{i=1}^n iX_i - \mu\right| > \delta\right) &\leq \sigma^2 \frac{2n(n+1)(2n+1)}{3n^2(n+1)^2\delta^2} \underset{n\to\infty}{\longrightarrow} 0 \\
\Rightarrow \frac{2}{n(n+1)}\sum_{i=1}^n iX_i &\underset{p}{\longrightarrow} \mu
\end{aligned}
$$

  (b) Show $\frac{6}{n(n+1)(2n+1)} \sum_{i=1}^n i^2 X_i \underset{p}{\rightarrow} \mu$

$$
\begin{aligned}
E\left(\frac{6}{n(n+1)(2n+1)}\sum_{i=1}^n i^2X_i\right) &= \frac{6}{n(n+1)(2n+1)}\sum_{i=1}^n i^2 \cdot E(X_i) \\
&= \mu \cdot \frac{6}{n(n+1)(2n+1)}\sum_{i=1}^n i^2 \\
&= \mu \\
0 \leq Var\left(\frac{6}{n(n+1)(2n+1)}\sum_{i=1}^n i^2X_i\right) &= \frac{36}{n^2(n+1)^2(2n+1)^2}\sum_{i=1}^n i^4 \cdot Var(X_i) \\
&= \sigma^2 \frac{36}{n^2(n+1)^2(2n+1)^2}\sum_{i=1}^n i^4 \\
&\leq \sigma^2 \frac{36}{n^2(n+1)^2(2n+1)^2}\sum_{i=1}^n n^4
\end{aligned}
$$

$$
\begin{aligned}
&= \sigma^2 \frac{36n^5}{n^2(n+1)^2(2n+1)^2} \\
&\underset{n\to\infty}{\longrightarrow} 0
\end{aligned}
$$

So, since we sandwiched the variance between zero and zero, it must go to zero in the limit. Applying Chebyshev,

$$
\begin{aligned}
P\left(\left|\frac{6}{n(n+1)(2n+1)}\sum_{i=1}^{n} i^2 X_i - \mu\right| > \delta\right) &\leq \frac{Var\left(\frac{6}{n(n+1)(2n+1)}\sum_{i=1}^{n} i^2 X_i\right)}{\delta^2} \underset{n\to\infty}{\longrightarrow} 0 \\
\Rightarrow \frac{6}{n(n+1)(2n+1)}\sum_{i=1}^{n} i^2 X_i &\underset{p}{\longrightarrow} \mu
\end{aligned}
$$

- 6.2.2
  Let X_i be independent r.v.'s

$$
P(X_i) = i = P(X_i = -i) = \frac{1}{2}
$$

Show that $\sum_{i=1}^{n} \frac{X_i}{n} \underset{p}{\not\to} 0$.

First, some notation. Let $S_n = \sum_{i=1}^{n} X_i$, and notice that S_n is symmetric, $P(S_n \geq 0) = P(S_n \leq 0) = p_n \geq \frac{1}{2}$

$$
\begin{aligned}
P\left(\left|\sum_{i=1}^{n} \frac{X_i}{n}\right| \geq 1\right) &= P\left(\left|\frac{S_n}{n}\right| \geq 1\right) \\
&= P(|S_n| \geq n) \\
&= P(S_n \leq -n) + P(S_n \geq n) \\
&= P(S_{n-1} \leq 0)P(X_n = -n) + P(S_{n-1} \geq 0)P(X_n = n) \\
&\quad \text{Since } S_{n-1} \text{ is independent of } X_n \\
&= \frac{p_{n-1}}{2} + \frac{p_{n-1}}{2} \\
&= p_{n-1} \geq \frac{1}{2} \ \forall \ n \\
&\underset{n\to\infty}{\not\to} 0
\end{aligned}
$$

Therefore, by definition, it does not converge in probability.

- 6.2.5
  Let X_i be iid $U[0,\theta]$ random variables, $\theta > 0$. Let $Y_n = \underset{i\in 1:n}{max}(X_i)$. Show $Y_i \underset{p}{\rightarrow} 0$

$$
\begin{aligned}
P(Y_n < k) &= \left(\frac{k}{\theta}\right)^n 1_{k\in[0,\theta]} \\
lim_{n\to\infty} \ F_{Y_n}(y) &= lim_{n\to\infty} \left(\frac{y}{\theta}\right)^n 1_{y\in[0,\theta]} + 1_{[y>\theta]} \\
&= 1_{[y\geq\theta]}
\end{aligned}
$$

By Corollary 6.2.25, since $F_{Y_n} \underset{n\to\infty}{\rightarrow} F_\theta = 1_{[y\geq\theta]}$, we know $Y_n \underset{p}{\rightarrow} \theta$.

- 6.2.11
  Let $X_1, X_2, ...$ be a sequence of iid r.v.'s each $\sim \mathcal{U}(0,1)$. For the geometric mean $G_n = (X_1 X_2 \cdots X_n)^{\frac{1}{n}}$ show that G_n converges in probability to c for some finite number c. Find c.

  To use any of the inequalities we've learned in chapter six, first we need to find the expected value.
  Note: X_i's are independent, so $E(X_i X_j) = E(X_i)E(X_j)$.

$$\begin{aligned} E(G_n) &= E(X_1^{\frac{1}{n}}) \cdots E(X_n^{\frac{1}{n}}) \\ &= \int_0^1 x_1^{\frac{1}{n}} dx \cdots \int_0^1 x_n^{\frac{1}{n}} dx = \left(\frac{1}{1+\frac{1}{n}}\right)^n \end{aligned}$$

  Now we just need to check on the variance to apply Chebyshev's inequality.

$$\begin{aligned} Var(G_n) &= E(G_n^2) - (E(G_n))^2 \\ E(G_n^2) &= \left(\frac{1}{1+\frac{2}{n}}\right)^n \quad \text{with same procedure as above} \\ \Rightarrow lim_{n\to\infty} Var(G_n) &= \frac{1}{e^2} - (\frac{1}{e})^2 = 0 \end{aligned}$$

  Now using Chebyshev's inequality with the definition of convergence in probability:
  Recall that $lim_{n\to\infty}\ (1+\frac{a}{n})^n = e^a$ (page 310) $\Rightarrow lim_{n\to\infty}\ E(G_n) = \frac{1}{e}$.

$$\begin{aligned} & \quad \text{recall by Chebyshev,} \\ \frac{Var(G_n)}{\delta^2} &\geq P\left(\left|G_n - \left(\frac{1}{1+\frac{1}{n}}\right)^n\right| > \delta\right) \\ &\geq P\left(\left|G_n - \left(\frac{1}{1+\frac{1}{n}}\right)^n\right| > \delta\right) \\ &\geq P\left(|G_n - \frac{1}{e} + \epsilon_n| > \delta\right) \\ &\geq P\left(|G_n - \frac{1}{e}| - |\epsilon_n| > \delta\right) \\ &\geq P\left(|G_n - \frac{1}{e}| > \delta_n^*\right) \\ &\underset{n\to\infty}{\longrightarrow} 0\ \forall\delta \end{aligned}$$

Thus $c = e^{-1}$.

- 6.2.12
Let $X_1, X_2, ...$ be iid r.v.'s with finite mean μ and finite variance σ^2. Show that the sample mean converges in mean square to μ (i.e. $\sum_{i=1}^{n} \frac{X_i}{n} \xrightarrow{2} \mu$).
Recall the definition of convergence in mean square, this would mean $\lim_{n\to\infty} E[(\overline{X} - \mu)^2] = 0$. So first we'll calculate $E[(\overline{X} - \mu)^2]$
Note: $E(\overline{X}^2) = Var(\overline{X}) + (E(\overline{X}^2))^2$.

$$\begin{aligned} E[(\overline{X} - \mu)^2] &= E[\overline{X}^2 - 2\mu\overline{X} + \mu^2] \\ &= E(\overline{X}^2) - 2\mu E(\overline{X}) + \mu^2 \\ &= \frac{\sigma^2}{n} + \mu^2 - 2\mu^2 + \mu^2 \\ &= \frac{\sigma^2}{n} \quad \underset{n\to\infty}{\longrightarrow} 0 \end{aligned}$$

Thus $\overline{X} \xrightarrow{2} \mu$.

- 6.2.13
Let $X_1, X_2, ...$ be iid r.v.'s with $E(X_1^4) < \infty$ and $Var(X_1) = \sigma^2$. If we have s_n^2 be the sample variance (as in Example 6.2.11) and $s_n^{2*} = \frac{(n-1)}{n} s_n^2$. Show that both s_n^2 and s_n^{2*} converge in probability to σ^2.
Note: There's more than one way to complete this problem. In the interest of finishing this key in a timely fashion for use in preparation of the midterm, only one method will be shown at this time.

This method will use Chebyshev's inequality, thus we need to calculate $E(s_n^2)$ and $Var(s_n^2)$.

$$\begin{aligned} E(s_n^2) &= E\left(\frac{\sum_{i=1}^{n}(X_1 - \overline{X})^2}{n-1}\right) = \frac{1}{n-1}\sum_{i=1}^{n} E\left((X_i - \mu)^2 - (\overline{X} - \mu)^2\right) \\ &= \frac{1}{n-1}\sum_{i=1}^{n} \underbrace{E\left((X_1 - \mu)^2\right)}_{Var(X_i)} - \underbrace{E\left((\overline{X} - \mu)^2\right)}_{Var(\overline{X})} = \frac{n(\sigma^2 - \frac{\sigma^2}{n})}{n-1} = \sigma^2 \end{aligned}$$

$$\begin{aligned} Var(s_n^2) &= \frac{1}{(n-1)^2} Var\left(\sum_{i=1}^{n}(X_1 - \overline{X})^2\right) \\ &= \frac{1}{(n-1)^2}\sum_{i=1}^{n} \underbrace{Var(X_1 - \overline{X})^2}_{\text{Let this be } \Phi} \quad \text{since } X_i\text{'s are independent} \end{aligned}$$

$$
\begin{aligned}
\Phi &= E\left((X_i - \overline{X})^4\right) - \left(E(X_i - \overline{X})^2\right)^2 \\
&= E\left[X_i^4 - 4X_i^3\overline{X} + 6X_i^2\overline{X} - 4X_i\overline{X} + \overline{X}^4\right] - \left[E(X_i^2) - 2E(X_i)\overline{X} + \overline{X}^2\right]^2 \\
\text{with algebra:} &= E(X_i^4) - 4\overline{X}E(X_i^3) + 4\overline{X}^2E(X_i^2) - (E(X_i^2))^2 + 4\overline{X}E(X_i)E(X_i^2)^2
\end{aligned}
$$

This could be simplified further, but point is that this Φ is finite. We know this because each term is finite since $E(X_i^4) < \infty$, thus all lower moments must be $< \infty$, and we know Φ is > 0 since it is a variance.
Hence we have:

$$
\begin{aligned}
Var(s_n^2) &= \frac{1}{(n-1)^2}\sum_{i=1}^{n}\Phi \\
&= \frac{n \cdot \Phi}{(n-1)^2} \quad \underset{n\to\infty}{\longrightarrow} 0
\end{aligned}
$$

From here we can immediately apply Chebyshev's inequality and we're done.

$$
\begin{aligned}
P\left(\left|s_n^2 - \sigma^2\right| > \delta\right) &\leq \frac{n\Phi}{(n-1)^2\delta^2} \underset{n\to\infty}{\longrightarrow} 0 \\
\Longrightarrow s_n^2 &\underset{p}{\longrightarrow} \sigma^2
\end{aligned}
$$

Now for s_n^{2*} (note that s_n^{2*} is commonly expressed as $\hat{\sigma}^2$).

$$
\begin{aligned}
E(s_n^{2*}) &= \frac{n-1}{n}E(s_n^2) = \frac{n-1}{n}\sigma^2 \\
Var(s_n^{2*}) &= \frac{(n-1)^2}{n^2}Var(s_n^2) = \frac{(n-1)^2}{n^2} \cdot \frac{n}{(n-1)^2} \cdot \Phi \\
&= \frac{\Phi}{n} \quad \underset{n\to\infty}{\longrightarrow} 0
\end{aligned}
$$

Now plug back into Chebychev's inequality:

$$
\begin{aligned}
P\left(\left|s_n^{2*} - \frac{n-1}{n}\sigma^2\right| > \delta\right) &= P\left(\left|\frac{n-1}{n}s_n^2 - \frac{n-1}{n}\sigma^2\right| > \delta\right) = P\left(\left|s_n^2 - \sigma^2\right| > \frac{n}{n-1}\delta\right) \\
&\leq P\left(\left|s_n^2 - \sigma^2\right| > \delta\right) \leq \frac{n\Phi}{(n-1)^2\delta^2} \underset{n\to\infty}{\longrightarrow} 0 \\
\text{thus } s_n^{2*} &\underset{p}{\longrightarrow} \sigma^2
\end{aligned}
$$