Stat 512
HW 3 Solutions
October 24, 2003

- 5.2.6

  Since the question specifically asks us to use this method, it is the only correct way to do this problem. First, we have the algebra:

$$
\begin{aligned}
E(X) &= E(X) \\
E(X^2) &= E(X(X-1)) + E(X) \\
E(X^3) &= E(X(X-1)(X-2)) + 3E(X(X-1)) + E(X) \\
E(X^4) &= E(X(X-1)(X-2)(X-3)) + 6E(X(X-1)(X-2)) + 7E(X(X-1)) + E(X) \\
\mu &= E(X) \\
Var(X) &= E(X^2) - E(X)^2 \\
Sk(X) &= E((X-E(X))^3) = E(X^3) - 3E(X^2)E(X) + 3E(X)E(X)^2 - E(X)^3 \\
&= E(X^3) - 3E(X^2)E(X) + 2E(X)^3 \\
Kur(X) &= E((X-E(X))^4) = E(X^4) - 4E(X^3)E(X) + 6E(X^2)E(X)^2 - 4E(X)E(X)^3 + E(X)^4 \\
&= E((X-E(X))^4) = E(X^4) - 4E(X^3)E(X) + 6E(X^2)E(X)^2 - 3E(X)^4
\end{aligned}
$$

(a) If $X \sim B(n,p)$ then:

$$
\begin{aligned}
E(X) &= \sum_{k=0}^{n} k \begin{pmatrix} n \\ k \end{pmatrix} p^k (1-p)^{n-k} \\
&= \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} p^k (1-p)^{n-k} \\
&= np \sum_{j=0}^{n-1} \frac{n-1!}{j!(n-1-j)!} p^j (1-p)^{n-1-j} \\
&= np \\
E(X(X-1)) &= \sum_{k=0}^{n} k(k-1) \begin{pmatrix} n \\ k \end{pmatrix} p^k (1-p)^{n-k} \\
&= \sum_{k=2}^{n} \frac{n!}{(k-2)!(n-k)!} p^k (1-p)^{n-k} \\
&= n(n-1)p^2 \sum_{j=0}^{n-2} \begin{pmatrix} n-2 \\ j \end{pmatrix} p^j (1-p)^{n-2-j} \\
&= n(n-1)p^2 \\
E(X(X-1)(X-2)) &= \sum_{k=0}^{n} k(k-1)(k-2) \begin{pmatrix} n \\ k \end{pmatrix} p^k (1-p)^{n-k} \\
&= \sum_{k=3}^{n} \frac{n!}{(k-3)!(n-k)!} p^k (1-p)^{n-k}
\end{aligned}
$$

$$\begin{aligned}
&= n(n-1)(n-2)p^3 \sum_{j=0}^{n-2} \left( \begin{array}{c} n-3 \\ j \end{array} \right) p^j (1-p)^{n-3-j} \\
&= n(n-1)(n-2)p^3 \\
E(X(X-1)(X-2)(X-3)) &= \sum_{k=0}^{n} k(k-1)(k-2)(k-3) \left( \begin{array}{c} n \\ k \end{array} \right) p^k (1-p)^{n-k} \\
&= \sum_{k=4}^{n} \frac{n!}{(k-4)!(n-k)!} p^k (1-p)^{n-k} \\
&= n(n-1)(n-2)(n-3)p^4 \sum_{j=0}^{n-2} \left( \begin{array}{c} n-3 \\ j \end{array} \right) p^j (1-p)^{n-3-j} \\
&= n(n-1)(n-2)(n-3)p^4
\end{aligned}$$

From this, we obtain:

$$\begin{aligned}
E(X) &= np \\
E(X^2) &= np(np-p+1) \\
E(X^3) &= np(n^2p^2 - 3np^2 + 2p^2 + 3np - 3p + 1) \\
E(X^4) &= np(n^3p^3 - 6n^2p^3 + 11np^3 - 6p^3 + 6n^2p^2 - 18np^2 + 12p^2 + 7np - 7p + 1) \\
E[X] &= np \\
Var(X) &= np(1-p) \\
Sk(X) &= np(2p-1)(p-1) \\
Kur(X) &= np(1-p)(3p(1-p)(n-2)+1)
\end{aligned}$$

If you used the central moment divided by the constant, it is also right. Use mathematica or maple to clean up your algebra if you don't want to do it by hand, because I don't care to wade through what you have, to see if it matches up.

(b) $X \sim Pois(\mu)$

$$\begin{aligned}
E[X] &= \sum_{x=0}^{\infty} x \cdot \frac{\mu^x e^{-\mu}}{x!} = \mu \sum_{x=1}^{\infty} \frac{\mu^{x-1} e^{-\mu}}{(x-1)!} \\
(\text{let } j = x-1) &= \mu \underbrace{\sum_{j=0}^{\infty} \frac{\mu^j e^{-\mu}}{j!}}_{1} = \mu \\
E[X(X-1)] &= \sum_{x=0}^{\infty} x(x-1) \cdot \frac{\mu^x e^{-\mu}}{x!} = \lambda^2 \sum_{x=2}^{\infty} \frac{\mu^{x-2} e^{-\mu}}{(x-2)!} \\
(\text{let } j = x-2) &= \lambda^2 \underbrace{\sum_{j=0}^{\infty} \frac{\mu^j e^{-\mu}}{j!}}_{1} = \lambda^2
\end{aligned}$$

$$
\begin{aligned}
E[X(X-1)(X-2)] &= \sum_{x=0}^{\infty} x(x-1)(x-2)\cdot\frac{\mu^x e^{-\mu}}{x!} = \lambda^3 \sum_{x=3}^{\infty}\frac{\mu^{x-3}e^{-\mu}}{(x-3)!} \\
\text{(let } j = x-3\text{)} &= \lambda^3 \underbrace{\sum_{j=0}^{\infty}\frac{\mu^j e^{-\mu}}{j!}}_{1} = \lambda^3 \\
E[X(X-1)(X-2)(X-3)] &= \sum_{x=0}^{\infty} x(x-1)(x-2)(x-3)\cdot\frac{\mu^x e^{-\mu}}{x!} = \lambda^4 \sum_{x=4}^{\infty}\frac{\mu^{x-4}e^{-\mu}}{(x-4)!} \\
\text{(let } j = x-4\text{)} &= \mu^4 \underbrace{\sum_{j=0}^{\infty}\frac{\mu^j e^{-\mu}}{j!}}_{1} = \mu^4
\end{aligned}
$$

$$
\begin{aligned}
E[X] &= \mu \\
E[X^2] &= \mu^2+\mu \\
E[X^3] &= \mu^3+3\mu^2+\mu \\
E[X^4] &= \mu^4+6\mu^3+7\mu^2+\mu \\
E[X] &= \mu \\
Var(X) &= \mu \\
Sk(X) &= \mu \\
Kur(X) &= \mu(3\mu+1)
\end{aligned}
$$

- 5.2.12
  X is a hypergeometric r.v. Find E(X) and Var(X). We proceed as in problem 1:

$$
E[X] = \frac{\sum_{x=0}^{m} x\begin{pmatrix} a \\ x \end{pmatrix}\begin{pmatrix} N-a \\ m-x \end{pmatrix}}{\begin{pmatrix} N \\ m \end{pmatrix}}
$$

note that $\begin{pmatrix} a \\ x \end{pmatrix} = \frac{a}{x}\begin{pmatrix} a-1 \\ x-1 \end{pmatrix}$

$$
\begin{aligned}
&= \frac{\sum_{x=1}^{m} a\begin{pmatrix} a-1 \\ x-1 \end{pmatrix}\begin{pmatrix} (N-1)-(a-1) \\ (m-1)-(x-1) \end{pmatrix}}{\frac{N}{m}\begin{pmatrix} N-1 \\ m-1 \end{pmatrix}} \\
&= \frac{a\cdot m}{N}\underbrace{\frac{\sum_{j=0}^{m-1}\begin{pmatrix} a-1 \\ j \end{pmatrix}\begin{pmatrix} (N-1)-(a-1) \\ (m-1)-j \end{pmatrix}}{\begin{pmatrix} N-1 \\ m-1 \end{pmatrix}}}_{1}
\end{aligned}
$$

$$\begin{aligned}
&= \frac{a \cdot m}{N} \\
E[X(X-1)] &= \frac{\sum_{x=0}^{m} x(x-1) \binom{a}{x} \binom{N-a}{m-x}}{\binom{N}{m}} \\
&= \frac{\sum_{x=2}^{m} a(a-1) \binom{a-2}{x-2} \binom{(N-2)-(a-2)}{(m-2)-(x-2)}}{\frac{N(N-1)}{m(m-1)} \binom{N-2}{m-2}} \\
&= \frac{am(a-1)(m-1)}{N(N-1)} \underbrace{\frac{\sum_{j=0}^{m-2} \binom{a-2}{j} \binom{(N-2)-(a-2)}{(m-2)-j}}{\binom{N-2}{m-2}}}_{1} \\
&= \frac{am(a-1)(m-1)}{N(N-1)} \\
E[X^2] &= \frac{am((a-1)(m-1)+N-1)}{N(N-1)} \\
Var[X] &= \frac{am(N-a)(N-m)}{N^2(N-1)}
\end{aligned}$$

- 5.2.14
  Let $X \sim N(\mu, \sigma^2)$ and $Y = e^X$ (a lognormal r.v.). Find $E(Y)$ and $Var(Y)$.

(i)

$$\begin{aligned}
E(Y) &= \int_{-\infty}^{\infty} e^x \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-mu)^2}{\sigma^2}} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2-2x\sigma^2}{\sigma^2}} dx \\
\text{complete the square} \qquad &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left[\frac{(x-\mu)^2-2x\sigma^2+\mu^2+2\mu\sigma^2+\sigma^4-\mu^2-2\mu\sigma^2-\sigma^4}{\sigma^2}\right]} dx \\
&= e^{\mu+\frac{\sigma^2}{2}} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left[\frac{(x-(\mu+\sigma^2))^2}{\sigma^2}\right]} dx}_{1} = e^{\mu+\frac{\sigma^2}{2}}
\end{aligned}$$

(ii) Note: $Var(Y) = E(Y^2) - (E(Y))^2$, and we have $E(Y)$ from (i) above, so we just need to get $E(Y^2)$.

$$E(Y^2) = \int_{-\infty}^{\infty} e^{2x} \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-mu)^2}{\sigma^2}} dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\frac{(x-\mu)^2-4x\sigma^2}{\sigma^2}} dx$$

$$\begin{aligned} \text{complete the square again} \qquad &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left[\frac{(x-\mu)^2-4x\sigma^2+\mu^2+4\mu\sigma^2+4\sigma^4-\mu^2-4\mu\sigma^2-4\sigma^4}{\sigma^2}\right]} dx \\ &= e^{2\mu+2\sigma^2} \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left[\frac{(x-(\mu+\sigma^2))^2}{\sigma^2}\right]} dx}_{1} = e^{2\mu+2\sigma^2} \end{aligned}$$

Thus $Var(Y) = e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2}$.

- 5.2.16
  Suppose that $P(X \geq 0) = 1$, then show $\sqrt{E(X)} \geq E(\sqrt{X})$.

$$\begin{aligned} \sqrt{E(X)} &\geq E(\sqrt{X}) \\ E(X) &\geq E(\sqrt{X})^2 \\ E(X) - E(\sqrt{X})^2 &\geq 0 \\ E((\sqrt{X})^2) - E(\sqrt{X})^2 &\geq 0 \end{aligned}$$

  Now let $Y = \sqrt{X}$ and we have $Var(Y) \geq 0$ which is true for any r.v. $Y$.

- 5.2.18
  Let $X$ be a Weibull r.v. with p.d.f. $f_X(x) = \beta x^{\beta-1} e^{-x^\beta} \cdot 1_{[0<x<\infty]}$.

  (a) Find $E(X^\beta)$.
  This can be done with a variable transformation or substituion.

$$E(X^\beta) = \int_0^\infty X^\beta \cdot \beta x^{\beta-1} e^{-x^\beta} dx$$

  Let $w = x^\beta$, then $dw = \beta x^{\beta-1}$ and we can substitute in to the equation above to get:

$$E(w) = \int_0^\infty w \cdot e^{-w} dw$$

  This is recognized as a Gamma integrated over the entire sample space, and thus $= 1$. The same result can be obtained if a variable transformation $y = x^\beta$ is used.

  (b) For $E(X)$ and $E(X^2)$, the transformed distribution above can be used and manipulated with constants to form a gamma distribution.
  The condition necessary for $E(X)$ and $E(X^2)$ to exist is the same condition for the density to exist, $\beta > 0$.

- 5.2.20
  Find the mean of r.v. $F$ with an F-distribution through finding the mean of $E(X)$ and $E(\frac{1}{Y})$ where $E(F) = \frac{n_2}{n_1}E(X)E(\frac{1}{Y})$, $X \sim \chi^2_{n_1}(0)$, and $Y \sim \chi^2_{n_2}(0)$.
  First note that since $X$ and $Y$ are both $\chi^2$, that means they're both gamma as well ($\chi^2_p = \Gamma(\frac{p}{2}, 2)$ where 2 is the mean).
  Thus $E(X) = \frac{n_1}{2} \cdot 2 = n_1$.
  Now $E(\frac{1}{Y})$ is also found using the gamma distribution. In general, for $Y \sim \Gamma(\alpha, \beta)$:

$$\begin{aligned} E(\frac{1}{Y}) &= \int_0^\infty \frac{1}{y} \cdot \frac{y^{\alpha-1}}{\beta^\alpha \Gamma(\alpha)} e^{-\frac{y}{\beta}} dy = \int_0^\infty \frac{y^{(\alpha-1)-1}}{\beta^\alpha \Gamma(\alpha)} e^{-\frac{y}{\beta}} dy \\ &= \frac{\beta^{\alpha-1}\Gamma(\alpha-1)}{\beta^\alpha \Gamma(\alpha)} \underbrace{\int_0^\infty \frac{y^{(\alpha-1)-1}}{\beta^{\alpha-1}\Gamma(\alpha-1)} e^{-\frac{y}{\beta}} dy}_{1} = \frac{1}{\beta\,(\alpha-1)} \\ &= \frac{1}{2(\frac{n_2}{2}-1)} = \frac{1}{n_2-2} \qquad \text{in the present case} \end{aligned}$$

  Thus $E(F) = \frac{n_2}{n_1} \cdot n_1 \cdot \frac{1}{n_2-2} = \frac{n_2}{n_2-2}$.

- Supplemental Problem

$$f(M,S) = \begin{cases} (1, X_1) & \text{if } X_1 \le a \text{ or } X_1 \ge b \\ (2, X_1 + X_2) & \text{otherwise} \end{cases}$$

  (a) Find the density for $(M, S)$. Let $f_1$ be the density of a $N(\mu, \sigma^2)$ RV

$$\begin{aligned} f(M,S) &= f(S, M=1)1_{m=1} + f(S, M=2)1_{m=2} \\ &= f_1(s)1_{m=1} + \int_a^b f_1(x)f_1(s-x)\,dx \cdot 1_{m=2} \end{aligned}$$

  (b) Let $\mu = 0$, $\sigma^2 = 1$, $a = 0$, $b = 2.7897$.

    i. Find $P[M = 1, S \le a]$.

$$= P(X_1 \le 0) = 1/2 \qquad (\text{Since } X_1 \sim N(0,1))$$

    ii. Find $P[M = 1, S \ge b]$.

$$= P(X_1 \ge 2.7897) = 1 - \Phi(2.7897) \approx 0.002638$$

iii. Find $P[M = 2, S \leq 0]$.

$$
\begin{aligned}
P(M = 2, S \leq 0) &= \int_{-\infty}^{0} \int_{0}^{2.7897} f_1(x) f_1(s-x)\, dx\, ds \\
&= \int_{0}^{2.7897} f_1(x) \int_{-\infty}^{0} f_1(s-x)\, ds\, dx \\
&\text{Notice } S \sim N(x, 1) \text{, and let } z = s - x. \\
&= \int_{0}^{2.7897} f_1(x) \int_{-\infty}^{-x} f_1(z)\, dz\, dx \\
&= \int_{0}^{2.7897} f_1(x) F_1(-x)\, dx \\
&= \int_{0}^{2.7897} f_1(x)(1 - F_1(x))\, dx \\
&= \int_{F_1(0)}^{F_1(2.7897)} 1 - y\, dy \\
&\approx 0.125
\end{aligned}
$$

iv. Find the value c such that $P[M = 1, S \geq b] + P[M = 2, S \geq c] = 0.025$.
I did not see a way to do this in closed form, so I used Mathematica. Including the code here, I had:

```
NIntegrate[Exp[-x^2/2 - (s - x)^2/2]/(2*Pi), {x, 0, 2.7897}, {s, c, Infinity}]
```

Varying values of c, we obtain that c = 2.79 gives 0.0223502, which is close enough to 1-(ii) for government work. If you're obsessive, 2.78969 gives as close as Mathematica and S will show.

(c) Derive a formula for $E(\hat{\mu})$ in the general case.

$$
\begin{aligned}
E(\hat{\mu}) &= \int_{-\infty}^{a} s \cdot f_1(s)\, ds + \int_{b}^{\infty} s \cdot f_1(s)\, ds + \int_{-\infty}^{\infty} \frac{s}{2} \int_{a}^{b} f_1(x) f_1(s-x)\, dx\, ds \\
&= \int_{-\infty}^{\infty} s \cdot f_1(s)\, ds - \int_{a}^{b} s \cdot f_1(s)\, ds + \frac{1}{2} \int_{a}^{b} f_1(x) \int_{-\infty}^{\infty} (s-x) f_1(s-x) + x f_1(s-x)\, ds\, dx \\
&= \mu - \int_{a}^{b} s \cdot f_1(s)\, ds + \frac{1}{2} \int_{a}^{b} f_1(x)(\mu + x)\, dx \\
&= \mu - \frac{1}{2} \left( \int_{a}^{b} s \cdot f_1(s)\, ds - \mu \int_{a}^{b} f_1(x)\, dx \right) \\
&= \mu - \frac{1}{2} \int_{a}^{b} (x - \mu) f_1(x)\, dx \\
&= \mu - \frac{1}{2} \int_{a-\mu}^{b-\mu} x f_2(x)\, dx
\end{aligned}
$$

Where $f_2$ is the density of a $N(0, \sigma^2)$ RV. It is obvious from this paramaterization that $a - \mu = -(b - \mu)$ in order for the second term to be zero, since $x \cdot f_2(x)$ is an odd function.