Biost/Stat 533
Emerson, Spr 09
Homework #1 Key
March 30, 2009

Written problems to be handed in Monday, April 6.

1. The χ^2, t, and F distributions are important "sampling distributions" commonly used in statistical inference. These distributions are derived as the exact distribution of certain statistics computed on normally distributed data. We are often ultimately interested the distribution-free interpretation of these statistics.

   a. Rigorously show that as n becomes large, a normal distribution provides a good approximation to the χ^2_n distribution. Make clear the parameters of the normal distribution, as well as the sense in which the approximation is valid.

   Ans: First note that by the definition of the chi squared distribution, if a random variable X_n is distributed χ^2_n, then it has the same distribution as $\sum_{i=1}^n Y_i$, where Y_i are independent χ^2_1 random variables, $i = 1, \ldots, n$. Then we know that each Y_i has mean 1 and variance 2. Thus, by the central limit theorem, for a sequence of chi squared random variables $\{X_n\}_{n=1}^\infty$ where X_n has n degrees of freedom and a sequence of independent chi squared random variables $\{Y_i\}_{1=1}^\infty$ with $Y_i \sim \chi^2_1$ we have

$$\sqrt{n}\left(\frac{1}{n}X_n - 1\right) \sim \sqrt{n}\left(\frac{1}{n}\sum_{i=1}^n Y_i - 1\right) \rightarrow_d \mathcal{N}(0,2).$$

   Thus for large n, the distribution of X_n is approximated by a normal distribution with mean n and variance $2n$ in the sense that for any $\varepsilon > 0$ there exists an N_ε such that for all $n \geq N_\varepsilon$

$$\left| Pr(X_n \leq x) - \Phi\left(\frac{x-n}{\sqrt{2n}}\right)\right| < \varepsilon$$

   where $\Phi(\cdot)$ is the cdf the standard normal distribution.

   b. Rigorously show that as n becomes large, a normal distribution provides a good approximation to the t_n distribution. Make clear the parameters of the normal distribution, as well as the sense in which the approximation is valid.

   Ans: The definition of the t distribution with n degrees of freedom is the distribution of a standard normal random variable divided by the square root of an independent chi squared random variable with n degrees of freedom that has itself been divided by its degrees of freedom. So letting $\{T_n\}_{n=1}^\infty$ be a sequence of t distributed random variables such that T_n has n degrees of freedom, Z be a standard normal random variable, and $\{X_n\}_{n=1}^\infty$ be a sequence of chi squared random variables where X_n has n degrees of freedom and each X_n is independent of Z. Now from part a we know

$$\sqrt{n}\left(\frac{1}{n}X_n - 1\right) \rightarrow_d N(0,2),$$

   hence we then know $\frac{1}{n}X_n \rightarrow_p 1$ and (via Mann-Wald) that $(\sqrt{\frac{1}{n}X_n})^{-1} \rightarrow_p 1$. And since $Z \sim \mathcal{N}(0,1)$, we also know $Z \rightarrow_d \mathcal{N}(0,1)$. Hence we can use Slutsky's theorem to show

$$T_n \sim \frac{Z}{\sqrt{X_n/n}} \rightarrow_d \mathcal{N}(0,1).$$

Thus for large n, the distribution of T_n is approximated by a standard normal distribution in the sense that for any $\varepsilon > 0$ there exists an N_ε such that for all $n \geq N_\varepsilon$

$$|Pr(T_n \leq t) - \Phi(t)| < \varepsilon$$

where $\Phi(\cdot)$ is the cdf the standard normal distribution.

c. Rigorously show that as n becomes large, a χ^2 distribution provides a good approximation to the $F_{m,n}$ distribution. Make clear the parameters of the χ^2 distribution, as well as the sense in which the approximation is valid.

Ans: The definition of the F distribution with m and n degrees of freedom is the distribution of the ratio of a chi squared random variable with m degrees of freedom that has been divided by its degrees of freedom and another, independent chi squared random variable with n degrees of freedom that has been divided by its degrees of freedom. So letting $\{F_n\}_{n=1}^{\infty}$ be a sequence of F distributed random variables such that F_n has m and n degrees of freedom, Y be a chi squared random variable with m degrees of freedom, and $\{X_n\}_{n=1}^{\infty}$ be a sequence of chi squared random variables where X_n has n degrees of freedom and each X_n is independent of Y. Now from part b we know that $(\sqrt{\frac{1}{n}X_n})^{-1} \rightarrow_p 1$. And since $Y \sim \chi^2_m$, we also know $Y \rightarrow_d \chi^2_m$. Hence we can use Slutsky's theorem to show

$$F_n \sim \frac{Y}{\sqrt{X_n/n}} \rightarrow_d \chi^2_m.$$

Thus for large n, the distribution of F_n is approximated by a chi squared distribution with m degrees of freedom in the sense that for any $\varepsilon > 0$ there exists an N_ε such that for all $n \geq N_\varepsilon$

$$\left|Pr(F_n \leq y) - Pr(\chi^2_m \leq y)\right| < \varepsilon.$$

2. The vast majority of statistics used for estimation, inference, and prediction involve a sum of some (possibly transformed) observations, with most such cases ultimately concerned with the differences in the means across groups. Suppose we have two groups of interest. Further suppose we can observe n (possibly transformed) independent random variables under a sampling scheme such that $Y_{0j} \sim (\mu_0, \sigma_0^2 < \infty)$ for $j = 1, \ldots, n_0$ and $Y_{1j} \sim (\mu_1, \sigma_1^2 < \infty)$ for $j = 1, \ldots, n_1 = n - n_0$. Let $\overline{Y}_{i\cdot} = \sum_{j=1}^{n_i} Y_{ij}/n_i$.

a. Under the restriction of a fixed value for n and the probability model described above, what is the optimal choice of n_0 to have maximal precision in an unbiased, distribution-free estimate $\hat{\theta}$ of $\theta = \mu_1 - \mu_0$?

Ans: An unbiased, distribution-free estimate $\hat{\theta}$ of $\theta = \mu_1 - \mu_0$ is based on sample means

$$\hat{\theta} = \overline{Y}_{1\cdot} - \overline{Y}_{0\cdot}$$

with distribution

$$\hat{\theta} \sim \left(\theta, \frac{\sigma_1^2}{n_1} + \frac{\sigma_0^2}{n_0}\right).$$

We try to find the value that minimizes the variance of this estimator for a constant $n = n_0 + n_1$

$$\frac{\partial}{\partial n_0}\left(\frac{\sigma_1^2}{n - n_0} + \frac{\sigma_0^2}{n_0}\right) = \frac{\sigma_1^2}{(n - n_0)^2} - \frac{\sigma_0^2}{n_0^2}$$

which when set equal to 0 yields

$$n_0 = \frac{\sigma_0}{\sigma_0 + \sigma_1} n \quad \text{and} \quad \frac{n_1}{n_0} = \frac{\sigma_1}{\sigma_0}.$$

b. Suppose we are able to randomly assign each subject to one of the two groups. Further suppose that we choose to randomize n subjects in an optimal ratio of r subjects in group 1 to 1 subject in group 0. We consider two possible randomization strategies:

   - "complete randomization" in which a biased coin in tossed for each subject to determine whether or not the subject is in group 1, and
   - "blocked randomization" subjects are sequentially assigned to treatment groups according to a random permutation of n_0 0's and n_1 1's, where $n_1/n_0 = r$ and $n_1 + n_0 = n$.

   Show that the unconditonal relative efficiency of $\hat{\theta}$ under blocked randomization compared to complete randomization is infinite.

   Ans: Under blocked randomization the variance is derived from above

$$\left(\frac{\sigma_1^2}{n_1} + \frac{\sigma_0^2}{n_0}\right) = \left((r+1)\frac{\sigma_0^2}{n_0}\right)$$

   Under complete randomization, we compute the variance using the (very important) formula

$$Var(Z) = Var_W(E[Z|W]) + E_W[Var(Z|W)]$$

   where we are interested in

$$Var(\hat{\theta}) = Var_{n_0}(E[\hat{\theta}|n_0]) + E_{n_0}[Var(\hat{\theta}|n_0)].$$

   Now $\hat{\theta}$ is unbiased for (almost) all choices of n_0, so we only need focus on the expectation of the conditional variance. Under complete randomization,

$$n_0 \sim \mathcal{B}(n, p = 1/(1+r)),$$

   so the expectation of the conditional variance is

$$E_{n_0}[Var(\hat{\theta}|n_0)] = \sum_{n_0=0}^{n} \binom{n}{n_0} \left(\frac{1}{r+1}\right)^{n_0} \left(\frac{r}{r+1}\right)^{n-n_0} \left(\frac{\sigma_1^2}{n-n_0} + \frac{\sigma_0^2}{n_0}\right).$$

   Clearly that variance is infinite, because there is positive probability placed on the possible choices of $n_0 = 0$ and $n_0 = n$.

c. After placing reasonable restrictions on the complete randomization strategy to avoid the infinite relative efficiency, explore the gains in unconditional efficiency of blocking for selected choices of n (say n = 20, 50, 100, 200, 500, and 1000) and r (say r = 1, 2, 3, 5, 10).

   Ans: So we ignore the very small possibility that $n_0 = 0$ or n. The S+ code attached to the end of this key shows the way I did this. From the results contained therein, you can see that the loss of efficiency is about 6% in small samples and negligible in large samples. Later we will be interested in blocked randomization when the variable being used to block the data is itself predictive of the outcome.

3. Suppose Y_i, $i = 1, 2, \ldots$ are independent, identically distributed random variables having mean μ, variance σ^2, and finite third and fourth central moments ω and ζ, respectively. Find the asymptotic joint distribution of the sample mean and sample variance.

   Ans: Let $m_1 = n^{-1}\sum_{i=1}^{n}(Y_i - \mu)$ and $m_2 = n^{-1}\sum_{i=1}^{n}(Y_i - \mu)^2$. Now

$$E\left[\begin{pmatrix} Y_i - \mu \\ (Y_i - \mu)^2 \end{pmatrix}\right] = \begin{pmatrix} 0 \\ \sigma^2 \end{pmatrix}$$

and

$$\Sigma = Cov \begin{pmatrix} Y_i - \mu \\ (Y_i - \mu)^2 \end{pmatrix} = \begin{pmatrix} \sigma^2 & \omega \\ \omega & \zeta - \sigma^4 \end{pmatrix}.$$

By the (multivariate) central limit theorem,

$$\sqrt{n} \begin{pmatrix} m_1 - 0 \\ m_2 - \sigma^2 \end{pmatrix} \rightarrow_d \mathcal{N}_2(0, \Sigma).$$

Now take $g((x,y)^T) = (x, y - x^2)^T$ and we will apply the delta method.

$$\nabla g((x,y)^T) = \begin{pmatrix} 1 & 0 \\ -2x & 1 \end{pmatrix}, \ \nabla g((0,\sigma^2)^T) = \mathbf{I}_2,$$

where $\mathbf{I}_2$ is the 2×2 identity matrix. Therefore,

$$\begin{aligned} \sqrt{n}(g(m_1, m_2) - g(0, \sigma^2)) &= \sqrt{n} \begin{pmatrix} \overline{Y} - \mu \\ n^{-1} \sum_{i=1}^n (Y_i - \mu)^2 - (\overline{Y} - \mu)^2 - \sigma^2 \end{pmatrix} \\ &= \sqrt{n} \begin{pmatrix} \overline{Y} - \mu \\ n^{-1} \sum_{i=1}^n (Y_i - \overline{Y})^2 - \sigma^2 \end{pmatrix} \\ &= \sqrt{n} \begin{pmatrix} \overline{Y} - \mu \\ \frac{n}{n-1} n^{-1} \sum_{i=1}^n (Y_i - \overline{Y})^2 - \sigma^2 \end{pmatrix} \rightarrow_d N_2(0, \Sigma) \\ &, \end{aligned}$$

by the delta-method and Slutsky's theorem, since $n/(n-1) \rightarrow_p 1$.

4. Consider again the probability model of problem 2 with n_0 and n_1 satisfying $n = n_0 + n_1$ and $r = n_1/n_0$ (not necessarily optimal).

   Ans: Some general notation.

   Let $\overline{Y}_0 = \sum_{i=1}^{n_0} Y_{0i}/n_0$ and $\overline{Y}_1 = \sum_{i=1}^{n_1} Y_{1i}/n_1$ denote the sample means for each group, and let $s_0^2 = \sum_{i=1}^{n_0} (Y_{0i} - \overline{Y}_0)^2/(n_0 - 1)$ and $s_1^2 = \sum_{i=1}^{n_1} (Y_{1i} - \overline{Y}_1)^2/(n_1 - 1)$ denote the sample variances for each group, with pooled variance estimate $s_p^2 = ((n_0 - 1)s_0^2 + (n_1 - 1)s_1^2)/(n_0 + n_1 - 2)$. Suppose further that as $n_0 \rightarrow \infty$ and $n_1 \rightarrow \infty$, $n_0/(n_0 + n_1) = \lambda = 1/(r+1)$ for some known constant $0 < \lambda < 1$. We are interested in making inference about $\theta = \mu_1 - \mu_0$.

   Under the assumption of equal variances (so $\sigma_0^2 = \sigma_1^2$), the usual statistical inference for this problem is based on the two sample t test for independent samples presuming equal variances using test statistic $T_e = (\overline{Y}_1 - \overline{Y}_0)/(s_p\sqrt{1/n_0 + 1/n_1})$ and assuming that T_e is distributed according to the t distribution with $n_0 + n_1 - 2$ degrees of freedom under the null hypothesis $H_0 : \delta = 0$. A confidence interval is constructed by inverting that test statistic.

   Under the assumption of unequal variances (so $\sigma_0^2 \neq \sigma_1^2$), the usual statistical inference for this problem is based on the two sample t test for independent samples allowing unequal variances using test statistic $T_u = (\overline{Y}_1 - \overline{Y}_0)/\sqrt{s_0^2/n_0 + s_1^2/n_1}$ and assuming that T_u is distributed according to the t distribution with k degrees of freedom under the null hypothesis $H_0 : \delta = 0$, where k might be determined by the Satterthwaite or Aspin-Welch approximations. A confidence interval is constructed by inverting that test statistic.

   a. Find the asymptotic distribution (as $n \rightarrow \infty$) of the t statistic that presumes equal variances.

   Ans: It is easiest in the long run if I solve for some general results assuming unequal variances first, and then at a later stage consider equal variances. Without loss of generality, assume $r \geq 1$. Define

$$D_i = \sum_{j=ri+1}^{r(i+1)} \frac{Y_{1j}}{r} - Y_{0i}.$$

then the D_i are independent, identically distributed random variables with

$$D_i \sim \left(\mu_1 - \mu_0, \frac{\sigma_1^2}{r} + \sigma_0^2\right).$$

So for $\overline{D}_n = \sum_{i=1}^{n_0} D_i/n_0$, the Levy CLT says

$$\sqrt{n_0}\left(\overline{D} - (\mu_1 - \mu_0)\right) \to_d \mathcal{N}\left(0, \frac{\sigma_1^2}{r} + \sigma_0^2\right),$$

and since $n_0/n \to \lambda = 1/(1+r)$

$$\sqrt{n}W_n \equiv \sqrt{(n)}(\overline{D}_n - (\mu_1 - \mu_0)) = \sqrt{n}(\overline{Y}_1 - \overline{Y}_0 - (\mu_1 - \mu_0)) \to_d \mathcal{N}(0, \frac{\sigma_0^2}{\lambda} + \frac{\sigma_1^2}{1-\lambda}) \quad (4.1)$$

Now by WLLN we know for $j = 0, 1$

$$\frac{1}{n_j}\sum_{i=1}^{n_j}(Y_{ji} - \mu_j)^2 \to_p \sigma_j^2$$

We also have

$$\frac{1}{n_j}\sum_{i=1}^{n_j}(Y_{ji} - \mu_j)^2 = \frac{1}{n_j}\sum_{i=1}^{n_j}(Y_{ji} - \overline{Y}_j)^2 + (\overline{Y}_j - \mu_j)^2$$

and by WLLN $\overline{Y}_j \to_p \mu_j$, so $(\overline{Y}_j - \mu_j)^2 \to_p 0$. Thus using Slutsky's along with the fact that $n/(n-1) \to_p 1$, we have

$$s_j^2 = \frac{1}{n_j - 1}\sum_{i=1}^{n_j}(Y_{ji} - \mu_j)^2 \to_p \sigma_j^2$$

Using this, we find

$$\sqrt{n}\sqrt{\frac{s_0^2}{n_0} + \frac{s_1^2}{n_1}} \to_p \sqrt{\frac{\sigma_0^2}{\lambda} + \frac{\sigma_1^2}{1-\lambda}} \quad (4.2)$$

and

$$s_p^2 = \frac{(n_0 - 1)s_0^2 + (n_1 - 1)s_1^2}{n-2} \to_p \lambda\sigma_0^2 + (1-\lambda)\sigma_1^2$$

so

$$\begin{aligned}\sqrt{n}s_p\sqrt{\frac{1}{n_0} + \frac{1}{n_1}} &= \sqrt{s_p^2(\frac{n}{n_0} + \frac{n}{n_1})} \\ &\to_p \sqrt{\frac{\sigma_0^2}{1-\lambda} + \frac{\sigma_1^2}{\lambda}}\end{aligned} \quad (4.3)$$

Now combining results (4.1) and (4.3), we find by Slutsky's that under the null hypothesis $H_0 : \mu_0 = \mu_1$

$$T_e = \frac{\sqrt{n}W_n}{\sqrt{n}s_p\sqrt{\frac{1}{n_0} + \frac{1}{n_1}}} \to_d \mathcal{N}\left(0, \frac{\frac{\sigma_0^2}{\lambda} + \frac{\sigma_1^2}{1-\lambda}}{\frac{\sigma_0^2}{1-\lambda} + \frac{\sigma_1^2}{\lambda}}\right). \quad (4.4)$$

Note: An alternative path to equation (4.1) could have used the univariate result based on the Levy CLT

$$\sqrt{n_0}(\overline{Y}_0 - \mu_0) \to_d \mathcal{N}(0, \sigma_0^2)$$
$$\sqrt{n_1}(\overline{Y}_1 - \mu_1) \to_d \mathcal{N}(0, \sigma_1^2)$$

and since $n_0/n \to \lambda$ and $n_1/n \to 1-\lambda$, by Slutsky's theorem we have

$$\sqrt{n}(\overline{Y}_0 - \mu_0) \to_d \mathcal{N}(0, \frac{\sigma_0^2}{\lambda})$$
$$\sqrt{n}(\overline{Y}_1 - \mu_1) \to_d \mathcal{N}(0, \frac{\sigma_1^2}{1-\lambda})$$

Then we need a result that would suggest that if $X_n \to_d X$ and $Y_n \to_d Y$ we would know that $X_n + Y_n \to_d X + Y$. This does not hold in general (consider the possibility that $X_n = -Y_n$, but X and Y are independent). It does seem logical that if X_n and Y_n are independent and X and Y are independent, that the result would hold, but the rigorous proof would most easily go through Skorokhod's theorem, which says that for a series of random variables X_n that converges in distribution to another random variable X, it is possible to find a series of random variables $X_n^* \sim X_n$ and random variable $X^* \sim X$ such that $X_n^* \to_{as} X^*$. Proofs about equality of distributions can be worked with X_n^* and X_* on the sets where they converge, and then inference brought back to the convergence in distribution, because of the equality of distributions.

b. Suppose we are interested in testing the strong null hypothesis $H_0 : Y_{1j} \sim Y_{0j}$. Under what conditions will is it asymptotically valid to make inference when assuming that the statistic in part (a) has a t distribution? When would such a procedure be asymptotically conservative? When would it be asymptotically anti-conservative?

Ans: Under the strong null hypothesis, the means will be equal <u>and</u> the variances will be equal. We find from (4.4) that when $\sigma_0^2 = \sigma_1^2 = \sigma^2$, $T_e \to_d \mathcal{N}(0,1)$. The usual inference for T_e is based on a t distribution with $n-2$ degrees of freedom, which asymptotically is equivalent to the standard normal (see problem 1), so the usual inference in this case is valid.

c. Suppose we are interested in testing the weak null hypothesis $H_0 : \mu_1 = \mu_0$. Under what conditions will is it asymptotically valid to make inference when assuming that the statistic in part (a) has a t distribution? When would such a procedure be asymptotically conservative? When would it be asymptotically anti-conservative?

Ans: Under the weak null hypothesis, there is no requirement that the variances be equal. When the group variances are not equal, we see from (4.4) that the asymptotic variance of T_e is 1 only if $\lambda = 1/2$. Otherwise, the asymptotic variance is

$$\frac{(1-\lambda)\sigma_0^2 + \lambda\sigma_1^2}{\lambda\sigma_0^2 + (1-\lambda)\sigma_1^2}$$

Now the inference will be conservative if the true variance is less than 1, and the inference will be anti-conservative if the true variance is greater than 1. We find that the true variance is greater than 1 precisely when

$$(1-2\lambda)\sigma_0^2 > (1-2\lambda)\sigma_1^2$$

This occurs when $\lambda < 0.5$ and $\sigma_0^2 > \sigma_1^2$ or when $\lambda > 0.5$ and $\sigma_0^2 < \sigma_1^2$. That is, the t test based on equal variances is asymptotically anti-conservative when the group with the largest variance has a smaller sample size than the group with the smallest variance. If we sample the group with the larger variance in greater numbers than the group with the smaller variance, the use of T_e leads to conservative inference (the true type I error is smaller than the nominal level, the coverage probability of confidence intervals is greater than the stated level, and the statistical power to detect an alternative hypothesis is diminished).

d. Find the asymptotic distribution (as $n \to \infty$) of the t statistic that allows unequal variances.

Ans: Combining results (4.1) and (4.2), we find by Slutsky's that under the null hypothesis $H_0 : \mu_0 = \mu_1$

$$T_u = \frac{\sqrt{n}W_n}{\sqrt{n}\sqrt{\frac{s_0^2}{n_0} + \frac{s_1^2}{n_1}}} \to_d \mathcal{N}\left(0, \frac{\frac{\sigma_0^2}{\lambda} + \frac{\sigma_1^2}{1-\lambda}}{\frac{\sigma_0^2}{\lambda} + \frac{\sigma_1^2}{1-\lambda}}\right) = \mathcal{N}(0,1) \qquad (4.5)$$

Note: For the t test that allows unequal variance, we could talk about the asymptotic distribution of the test statistic under the conditions that $\min(n_0, n_1) \to \infty$, because this statistic's asymptotic distribution did not depend on the ratio of sample sizes. The proof of this is most easily effected using the Lindeberg-Feller CLT and a proof similar to that for the asymptotics of the GLSE.

e. Suppose we are interested in testing the strong null hypothesis $H_0 : Y_{1j} \sim Y_{0j}$. Under what conditions will is it asymptotically valid to make inference when assuming that the statistic in part (d) has a t distribution? When would such a procedure be asymptotically conservative? When would it be asymptotically anti-conservative?

Ans: From (4.5) we find that T_u is asymptotically standard normal no matter whether the variances are equal or not. Hence, as the t with k degrees of freedom is asymptotically standard normal, and as the degrees of freedom for the Aspin-Welch test is bounded below by the smaller of $n_0 - 1$ and $n_1 - 1$, we know that the usual inference based on T_u is asymptotically valid.

f. Suppose we are interested in testing the weak null hypothesis $H_0 : \mu_1 = \mu_0$. Under what conditions will is it asymptotically valid to make inference when assuming that the statistic in part (d) has a t distribution? When would such a procedure be asymptotically conservative? When would it be asymptotically anti-conservative?

Ans: Same answer as part e.

Aside: What do the above results suggest about the validity of regression based on linear regression models in the presence of heteroscedasticity?

Ans: The t test with equal variances is the special case of unweighted linear regression in which the single predictor is a binary variable. Because standard software for the unweighted linear regression model produces variance estimates based on the assumption of equal error variance across all observations, the above results tell us that that inference will be incorrect in the absence of a balanced design.

```
#############
##
## Assuming the ratio is optimal each time
##
#############

RSLT <- NULL

for (n in c(20,50,100,200,500,1000)) {
   for (r in c(1,2,3,5,10)) {
      n0b <- round(n / (r + 1))
      n1b <- n - n0b

      n0 <- 1:(n-1)
      n1 <- n - n0
      p <- dbinom(n0,n,1/(1+r))
      p <- p / sum(p)
      v <- 1/n0 + r^2/n1
      CRDv <- sum(v*p)
      Blockv <- (1+r)^2 / n
      RSLT <- rbind(RSLT,c(n,r,CRDv,Blockv,CRDv / Blockv))
   }
}

tbl.opt <- matrix(RSLT[,5],6,5,byrow=T)
dimnames(tbl.opt) <- list(c(20,50,100,200,500,1000),c
   (1,2,3,5,10))
tbl.opt
            1        2        3        5       10
  20 1.059904 1.065232 1.069352 1.060195 1.029918
  50 1.021318 1.021866 1.022911 1.025824 1.028232
 100 1.010314 1.010429 1.010632 1.011139 1.012951
 200 1.005077 1.005103 1.005149 1.005256 1.005577
 500 1.002012 1.002016 1.002023 1.002039 1.002082
1000 1.001003 1.001004 1.001006 1.001010 1.001020

#############
##
## Assuming the variances are equal each time
##
#############

RSLT <- NULL
```

```
for (n in c(20,50,100,200,500,1000)) {
    for (r in c(1,2,3,5,10)) {
        n0b <- round(n / (r + 1))
        n1b <- n - n0b

        n0 <- 1:(n-1)
        n1 <- n - n0
        p <- dbinom(n0,n,1/(1+r))
        p <- p / sum(p)
        v <- 1/n0 + 1/n1
        CRDv <- sum(v*p)
        Blockv <- 1 / n0b + 1 / n1b
        RSLT <- rbind(RSLT,c(n,r,CRDv,Blockv,CRDv / Blockv))
    }
}

tbl.nonopt <- matrix(RSLT[,5],6,5,byrow=T)
dimnames(tbl.nonopt) <- list(c(20,50,100,200,500,1000),c
    (1,2,3,5,10))
tbl.nonopt
            1        2        3        5       10
  20 1.059904 1.128612 1.169877 1.137302 1.197966
  50 1.021318 1.043435 1.026240 1.076624 1.368528
 100 1.010314 1.010626 1.025115 1.064256 1.110250
 200 1.005077 1.010188 1.012086 1.014055 1.041773
 500 1.002012 1.004029 1.004732 1.005359 1.009835
1000 1.001003 1.001006 1.002349 1.005853 1.010205
```