

Biost 517
Applied Biostatistics I
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Lecture 5:
Descriptive Measures of Spread

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Lecture Outline
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- Univariate Measures of Spread
- Univariate Measures of Skewness
- Univariate Measures of Tendency to Extreme Values
- Depictions of Entire Distribution
- Example: Prognostic value of PSA

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**Univariate Measures
of Spread**
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Range
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- Definition varies:
 - Minimum, maximum values
 - Maximum - minimum is used by some people

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Range: Types of Variables

- Only makes sense for ordered variables
- Not appropriate for censored time to event
 - Instead use Kaplan-Meier curves to estimate survival or censoring distributions

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Range: Purpose

- Detecting errors in data collection, entry
 - Values out of range
- Materials and Methods
 - Limits of subjects in sample
- Less useful for quantifying or comparing distributions

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Range: Scientific Questions

- Scientific questions
 - Not useful unless range of possible values differs across populations
 - But even then, the sampling distribution of the min and max depends quite heavily on the sample size

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Minimum: Sampling Distribution

- Minimum of n independent and identically distributed random variables

$$\Pr(X_{(1)} \geq x) = (\Pr(X \geq x))^n$$

- Tends to estimate the $1/(n+1)$ -th quantile of the distribution of X
 - 25th %ile when $n = 3$
 - 1st %ile when $n = 99$

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Maximum: Sampling Distribution

- Maximum of n independent and identically distributed random variables

$$\Pr(X_{(n)} \leq x) = (\Pr(X \leq x))^n$$

- Tends to estimate the $n/(n+1)$ -th quantile of the distribution of X
 - 75th %ile when $n = 3$
 - 99th %ile when $n = 99$

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Interquartile Range (IQR)

- Definition varies:
 - 25th, 75th percentiles of sample
 - Difference between quartiles is used by some people

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IQR: Type of Variables

- Only makes sense for ordered variables
- Not appropriate for censored time to event
 - Estimate quantiles using Kaplan-Meier

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IQR: Purpose

- Materials and Methods: Characterizing the sample
 - A measure of spread less sensitive to outliers
 - Central 50% of the data
- Assessing validity of assumptions
 - Check for equal spread of distributions
 - BUT: most assumptions are about variances

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IQR: Scientific Uses

- Quantifying or comparing distributions
 - Sometimes we are scientifically interested in the spread of the distribution
 - The sample quartiles consistently estimate the population quartiles
 - Central tendency not based on sample size

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Variance

- Definition
 - The average squared distance from the mean

$$\text{population } \sigma^2 = \frac{1}{n} \sum_{i=1}^n (X_i - E(X))^2$$

$$\text{sample } s^2 = \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2$$

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Variance: Interpretation

- Population variance has theoretical basis as second central moment of distribution
 - Squared error as a measure of spread
 - Squaring accentuates errors
 - More convenient mathematically
- Relevance to sampling distributions used in statistical inference
 - Variance is a fundamental parameter of the normal distribution
 - Sample variance is an unbiased estimator

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Variance: Types of Variables

- Variance is a mean
 - Best used with numeric variables having interpretable differences
 - BUT it will be used whenever we are comparing means of distributions
 - Problematic with censored measurements
 - Times are mixture of times to event and times to censoring
 - Indicators of event are measured over varying times

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Variance: Purpose

- **Materials and Methods: Characterizing the spread of the distribution**
 - Larger variance means more variable measurements
 - But units are squared units of observations
 - E.g., variance of age is measured in years squared
 - And sensitive to outliers
- **Assessing validity of models**
 - Many analysis methods rely on assumptions about within group variances

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Variance: Scientific Questions

- **Quantifying or comparing distributions**
 - Sometimes we are scientifically interested in the spread of the distribution
 - As a mean, the sampling distribution of the variance in large samples is known
 - But it does take larger sample sizes than is required for inference about the mean

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Standard Deviation (SD)

- **Definition**
 - The square root of the variance

population $\sigma = \sqrt{\frac{1}{n} \sum_{i=1}^n (X_i - E(X))^2}$

sample $s = \sqrt{\frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2}$

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SD: Interpretation

- **Related to variance; same units as mean**
 - Sample SD is not unbiased for population SD
- **“Width” of the distribution**
 - For any distribution
 - At least 89% of data within 3 SD of mean
 - For the **normal (Gaussian)** distribution
 - About 2/3 of data is within 1 SD of mean
 - About 95% of data is within 2 SD of mean
 - About 99.7% of data is within 3 SD of mean

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SD vs Variance: Uses

- Variance is just the squared SD
 - Use SD descriptively
 - Units are the same as the measurements
 - Can evaluate equality of variances for assumptions
 - If SDs are equal, then so are variances
 - Use variance for inference
 - Mathematics and distributional theory is better defined for the variance
- SD used for standardization of statistics when making inference about means

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SD: Standardized Statistics

- Often we measure distance of data from mean in units of SD's
 - (sometimes called “normalized”, but does not guarantee normality of data)

$$X_1, \dots, X_n \sim (\text{mean } \mu, \text{variance } \sigma^2)$$

$$Z_i = \frac{X_i - \mu}{\sigma}$$

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Mean Deviation

- Definition
 - The average absolute distance from the mean

$$\frac{1}{n} \sum_{i=1}^n |X_i - \bar{X}|$$

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Mean Deviation: Uses

- By types of variables
 - Mean deviation is a mean
 - Best for numeric variables having interpretable differences and measured without censoring
- By purpose of descriptive statistics
 - Possible alternative to SD / variance, however
 - Absolute values are harder to work with in calculus
 - Sampling distribution is thus harder to derive
 - Not helpful in statistical inference

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Univariate Measures of Skewness

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Coefficient of Skewness

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- Definition
 - The average cubed distance from the mean divided by cube of the standard deviation

$$\text{sample } \frac{1}{(n - 1) s^3} \sum_{i=1}^n (X_i - \bar{X})^3$$

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Skewness: Interpretation

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- Cubing the distance from the mean
 - Accentuates outliers
 - Does allow positive outliers to cancel out negative outliers
- Symmetric distributions will have a skewness coefficient of 0

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Skewness: Types of variables

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- Skewness is a mean
 - Best used with numeric variables having interpretable differences
 - Not of interest with censored random variables

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Skewness: Purpose

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- Materials and Methods: Characterizing the distribution
 - Describing tendency to outliers (in one direction)
- Assessing validity of assumptions
 - Sometimes need symmetric distributions
 - Distributions with outliers generally require larger sample sizes for accurate inference
 - Outliers are sometimes too influential

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Other Measures of Skewness

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- I rarely (never?) compute the coefficient of skewness
 - Usually only interested in qualitatively describing tendency to large (or small) outlying values
- I just use other descriptive statistics to judge possibility of outliers
 - Mean, SD, min, p25, med, p50, max
 - Look at histogram if indicated

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Symmetric distributions

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- Properties of symmetric distributions
 - Mean is equal to the median
 - The median is midway between the minimum and maximum
 - The 25th and 75th percentiles are equidistant from the median

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Signs of Skewed Distributions

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- Descriptive statistics which suggest skewed distributions (especially when due to outliers)
 - The sample median is
 - markedly different from the sample mean
 - (Mean is greatly affected by outliers)
 - not midway between the minimum and maximum
 - not midway between the 25th and 75th percentiles

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Signs of Skewed Distributions

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- An additional criterion is based on the properties of the standard deviation
 - If about 2 standard deviations away from the mean includes impossible values, then it is often the case that large outliers exist
 - E.g., for measurements that must be positive, a standard deviation greater than one-half the mean may suggest the presence of outliers

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Univariate Measures of Tendency to Extreme Data

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Coefficient of Kurtosis

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- Definition of kurtosis
 - The average fourth power distance from the mean divided by the square of the variance
 - Coefficient of kurtosis subtracts 3

$$\frac{1}{(n - 1)s^4} \sum_{i=1}^n (X_i - \bar{X})^4 - 3$$

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Kurtosis: Interpretation

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- The fourth central moment will accentuate observations in the tail of the distribution
- In the normal distribution, the fourth central moment is $3\sigma^4$
 - Coefficient of kurtosis is 0

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Kurtosis: Types of Variables

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- Kurtosis is a mean
 - Best used with numeric variables having interpretable differences
 - (As usual,) variables measuring censored times to events can not be described appropriately using the sample coefficient of kurtosis

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Coefficient of Kurtosis: Uses

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- By purpose of descriptive statistics
 - Characterizing the distribution
 - Describing tendency to heavy tails
 - Assessing validity of assumptions
 - The heavier the tails, the larger the sample size needed before the central limit theorem is a reasonable approximation
 - BUT, heavy symmetric tails often shows up in samples as skewness
 - I just tend to look for skewness

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Characterizing the Entire Distribution

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General Comments

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Statistical Setting

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- Statistical analysis of a sample
 - Usually want to make inference to population
- Probability model
 - Describes variation observed in a population
 - We regard that the data were sampled from that probability model

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Probability Model Classification

- The type of measurement is either
 - Discrete: a countable number of different values are possible for the measurement, or
 - Continuous: an uncountable number of different values are possible for the measurement

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Samples vs Populations

- All (finite) samples are discrete
 - Descriptive measures appropriate for discrete measurements always make sense
- Often, however, we will be trying to estimate quantities that are only appropriate for continuous measurements (e.g., densities)

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Describing Discrete Distns

- Typically give probability of each outcome
 - Prob mass function (pmf): $\Pr(X = x)$ for each x
 - Explicit numbers
 - (only possibility for unordered categorical data)
 - Formulas
 - (especially for measurements representing counts)
- Ordered variables: we can also consider the cumulative distribution function (cdf):
 - $\Pr(X \leq x)$ for every x

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Describing Continuous Distns

- Typically give a function that can be used to define the probability that the random variable is in some interval
 - Density (pdf)
 - Similar to $\Pr(X = x)$
 - (but must be integrated over an interval)
 - Cumulative distribution function (cdf)
 - $\Pr(X \leq x)$ for every x
 - Survivor function
 - $\Pr(X > x)$ for every x

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Why Describe Entire Distribution

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- Assessing validity of data
 - Viewing outliers
- Quantifying distributions within groups
 - Simultaneously consider location, spread
- Assessing validity of assumptions for modeling
 - Outliers, shape of distribution
- Hypothesis generation
 - Multimodality, etc.

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Tabling a Distribution

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Frequency Table

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- Frequency or proportion for each possible value of a discrete variable
 - Not defined for a continuous measurement in a population, but always defined for a sample
 - With unordered categorical data this is the most logical summary
 - With a sample from a continuous variable, this makes most sense when data is grouped into intervals
 - E.g., age divided into decades

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Frequency Table: Issues

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- Missing data
 - Must consider whether missing data should be counted as part of the denominator or not
- Order of listing categories
 - Unordered data: alphabetical versus most frequent, etc.
 - Ordered data: usually use ordering

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Stata Ex: Bone Scan Score

- `tabulate bss`

bss	Freq.	Percent	Cum.
1	5	10.42	10.42
2	13	27.08	37.50
3	30	62.50	100.00
Total	48	100.00	

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Stata Ex: Bone Scan Score

- `tabulate bss, missing`

bss	Freq.	Percent	Cum.
1	5	10.00	10.00
2	13	26.00	36.00
3	30	60.00	96.00
.	2	4.00	100.00
Total	50	100.00	

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Frequency Table: Issues

- Categorization of continuous data
 - Number of groups
 - Tradeoffs between finer resolution and lots of groups with zero counts
 - Based on average counts per group?
 - » log base 2 of N?
 - Width of intervals
 - Cutpoints
 - Based on scientific interest
 - Based on number of observations
 - » E.g., intervals based on quintiles

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Categorizing Continuous Data

- My recommendations
 - Groups based on scientific considerations
 - E.g., Age by decades rather than quintiles
 - Number of groups
 - Reasonable sample sizes for reliable estimates
 - Space constraints in tables

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Stata: Categorizing Data

- Categorizing continuous random variables
 - “recode var rulelist”
 - Replaces values of the variable
 - Rules of the form (for variable x)
 - min/3=1 (changes $x \leq 3$ to 1)
 - 4=2 (changes 4 to 2)
 - 5/9=3 (changes $5 \leq x \leq 9$ to 3)
 - 10/max=4 (changes $x \geq 10$ to 4)
 - Each rule contains both endpoints of range, last rule listed is used to resolve conflicts
 - Labels can be assigned using “label”

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Stata Ex: Age

- tabulate age

age	Freq.	Percent	Cum.
58	2	4.00	4.00
61	5	10.00	14.00
62	1	2.00	16.00
63	5	10.00	26.00
64	5	10.00	36.00
65	2	4.00	40.00
66	6	12.00	52.00
68	7	14.00	66.00
69	3	6.00	72.00
70	2	4.00	76.00
71	4	8.00	84.00
73	1	2.00	86.00
74	1	2.00	88.00
75	2	4.00	92.00
78	1	2.00	94.00
79	1	2.00	96.00
81	1	2.00	98.00
86	1	2.00	100.00
Total	50	100.00	

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Stata Ex: Categorizing Age

- g agectg = age
- recode agectg min/60=1 60/70=2 70/80=3 80/max=4
- tabulate agectg

agectg	Freq.	Percent	Cum.
1	2	4.00	4.00
2	34	68.00	72.00
3	12	24.00	96.00
4	2	4.00	100.00
Total	50	100.00	

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Graphing a Distribution

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Stem-leaf Plot

- Stem-leaf plot : Intermediate to a tabular listing of the data and a histogram
 - Only appropriate for ordered quantitative data
 - Construction of the stem-leaf plot
 - Each measurement is divided into
 - a “stem” by truncating the observation in some position, e.g., tens digit
 - a “leaf” the remaining value
 - (thus the measurement is equal to the “stem”+“leaf”)

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Stem-leaf Plot

- Construction of the stem-leaf plot (cont.)
 - The rows are the ordered “stem”s
 - In each row, the first digit of all the “leaf”s for that stem are listed (ordered or unordered)
- Advantages
 - Graphical appearance of a histogram
 - Retains more information about the individual measurements
 - Can be made “on the fly”
- Stata Command: “stem”

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Stata Ex: Stem-Leaf Plot of Age

```
. stem age
      5. | 88
      6* | 11111
      6t | 233333
      6f | 4444455
      6s | 666666
      6. | 8888888999
      7* | 001111
      7t | 3
      7f | 455
      7s |
      7. | 89
      8* | 1
      8t |
      8f |
      8s | 6
```

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S-Plus: Stem-Leaf Plot of Age

```
Decimal point is 1 place to the right of the colon
5 : 88
6 : 1111123333344444
6 : 55666666888888999
7 : 001111134
7 : 5589
8 : 1

High: 86
```

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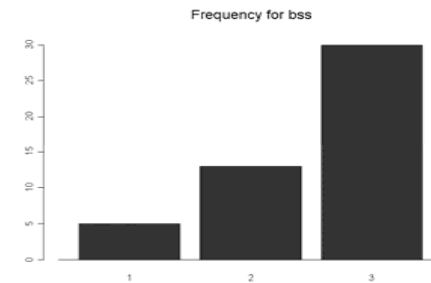
Bar Plots

- Frequencies for categorical data
 - A separate bar for each category
 - (Not exactly the same as a histogram, which is used to estimate a density)

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Ex: Bone Scan Score (S-Plus)

- Bar plot of bone scan score in PSA data



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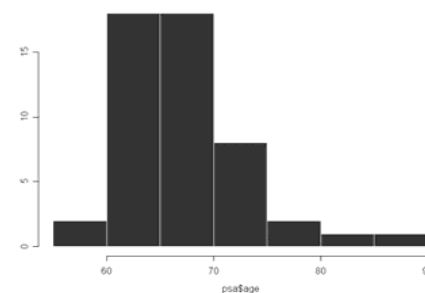
Histograms

- A plot of the frequency of (categorized) continuous measurements
 - Tabulate counts of each (grouped) measurement
 - Plot bars for each group
 - Width is width of interval
 - Height such that area is proportional to the count
 - (wider intervals should decrease the height accordingly)

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Ex: Histogram of Age (S-Plus)

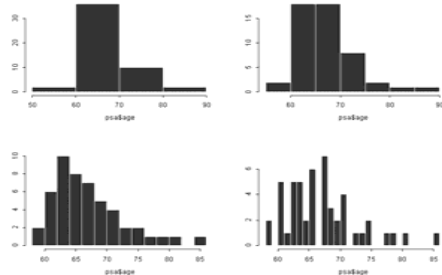
- Histogram of age in PSA dataset



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Ex: Histograms of Age (S-Plus)

- Histograms with varying number of groups can look quite different for the same data



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Histogram: Comments

- Histograms are attempting to estimate a density
- The appearance of a histogram can be quite variable depending upon the selection of the groups
 - Number of groups
 - Cutpoints of groups
- Density estimation is probably better

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Density Estimation

- Density estimates are essentially smoothed histograms
 - Only make sense for continuous measurements
- Smoothers can be used to provide better estimates of the density
 - In a kernel smoother, each point is “distributed” over a range of measurements
 - The “kernel” describes how many adjacent measurements are used to estimate the density and how the points are weighted

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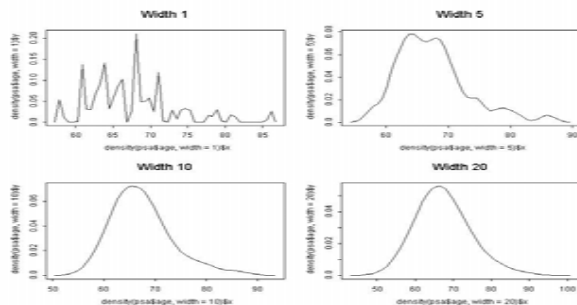
Stata: Density Estimates

- “`kdensity var, (options)`”
 - Options include
 - Shape of kernel (biweight, cosine, etc.)
 - Width of window
 - » how distant a point can influence density estimate
 - Number of points estimated
- (I tend to use default values for the options)

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Ex: Density Estimation

- Age from PSA data set (Gaussian window, varying widths)



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Cumulative Distribution, Survivor Graphs

- For ordered variables
 - Used to estimate the corresponding quantity for a population
 - These functions can sometimes be estimated (and graphed) for censored data (unlike histograms, densities, etc.)
- We will discuss these graphs further next lecture

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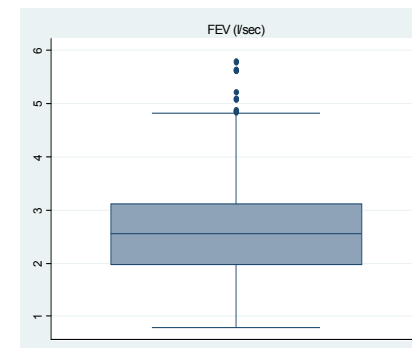
Box Plots

- Display several summary measures simultaneously
 - A box is drawn from the lower quartile to the upper quartile, with a dividing line drawn at the median
 - Whiskers are either
 - min and max (in the absence of “outliers”), or
 - limits of “nonoutlying” data (as defined by an arbitrary criterion)
 - “Outliers” are plotted separately

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Box Plots: FEV

- graph box fev, t1(“FEV(l/sec)”)



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Example: Use of Univariate Descriptive Statistics

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Standard Univariate Description

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- Often easier to just ask for standard descriptive statistics on all variables
 - Sample size
 - Number of missing
 - Mean
 - Standard Deviation
 - Minimum
 - 25th percentile
 - Median (50th percentile)
 - 75th percentile
 - Maximum

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Standard Univariate Description

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- We must then consider how to use the relevant statistics (and how to ignore the irrelevant ones)
 - Scientific relevance
 - Descriptive measures
 - Measures of location
 - Measures of spread
 - Detecting outliers

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Example:

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- Prognostic value of PSA in hormonally treated prostate cancer
 - Usefulness of nadir PSA in predicting time in remission
 - Type of question?
 - Comparing distributions?
 - Predicting time in remission?

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Example: PSA Data Variables

- Prognostic value of PSA in hormonally treated prostate cancer
 - ptid Patient ID
 - nadir Lowest PSA following treatment
 - pretx Pre-treatment PSA
 - ps Performance status (0 – 100)
 - bss Bone scan score (1, 2, or 3)
 - grade Tumor grade (1, 2, or 3)
 - age Age (years)
 - obstime Time until relapse or end of study (months)
 - inrem In remission at obstime

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Ex: PSA Descriptive Statistics

	<u>n</u>	<u>ms</u>	<u>mean</u>	<u>stdev</u>	<u>min</u>	<u>25%le</u>	<u>mdn</u>	<u>75%le</u>	<u>max</u>
ptid	50	0	25.5	14.6	1.0	13.2	25.5	37.8	50
nadir	50	0	16.4	39.2	0.1	0.2	1.0	9.5	183
pretx	50	7	670.8	1287.6	4.8	52.0	127.0	408.0	4797
ps	50	2	80.8	11.1	50.0	80.0	80.0	90.0	100
bss	50	2	2.5	0.7	1.0	2.0	3.0	3.0	3
grade	50	9	2.2	0.8	1.0	2.0	2.0	3.0	3
age	50	0	67.4	5.8	58.0	63.2	66.0	70.0	86
obstime	50	0	28.5	18.4	1.0	12.5	28.0	42.0	75
inrem	50	0	0.3	0.4	0.0	0.0	0.0	1.0	1

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Example: PSA Data Variables

- Types of data
 - ptid Unordered categorical (coded as numbers)
 - nadir Continuous (ratio)
 - pretx Continuous (ratio)
 - ps Continuous (ratio) (measured discretely)
 - bss Ordered categorical
 - grade Ordered categorical
 - age Continuous (ratio)
 - obstime Censored continuous
 - inrem Binary indicator of censoring for obstime

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Example: PSA Data Variables

- Relevant univariate statistics
 - ptid (Mode: are observations independent?)
 - nadir Mean, SD, Min, Max, Quantiles
 - pretx Mean, SD, Min, Max, Quantiles
 - ps Mean, SD, Min, Max, Quantiles
 - bss Min, Max, Quantiles (Frequencies)
 - grade Min, Max, Quantiles (Frequencies)
 - age Mean, SD, Min, Max, Quantiles
 - obstime (Kaplan-Meier estimates needed)
 - inrem (Kaplan-Meier estimates needed, though mean of inrem does tell us proportion of uncensored observations but not time)

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Example: Relevant Univariate Statistics

	<u>n</u>	<u>ms</u>	<u>mean</u>	<u>stdev</u>	<u>min</u>	<u>25%ile</u>	<u>mdn</u>	<u>75%ile</u>	<u>max</u>
ptid	50	0							
nadir	50	0	16.4	39.2	0.1	0.2	1.0	9.5	183
pretx	50	7	670.8	1287.6	4.8	52.0	127.0	408.0	4797
ps	50	2	80.8	11.1	50.0	80.0	80.0	90.0	100
bss	50	2			1.0	2.0	3.0	3.0	3
grade	50	9			1.0	2.0	2.0	3.0	3
age	50	0	67.4	5.8	58.0	63.2	66.0	70.0	86
obstime	50	0							
inrem	50	0							

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Example: PSA Data Skewness

- Detecting skewness
 - Both nadir and pretx appear markedly skewed
 - Mean, median markedly different
 - Median not midpoint of range
 - Median not midpoint of interquartile range
 - SD greater than one-half of mean for these positive measurements
 - No evidence of skewness (that I care about) for age or pss

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Ex: Bone Scan Score

- tabulate bss

bss	Freq.	Percent	Cum.
1	5	10.42	10.42
2	13	27.08	37.50
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Total	48	100.00	

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Ex: Bone Scan Score

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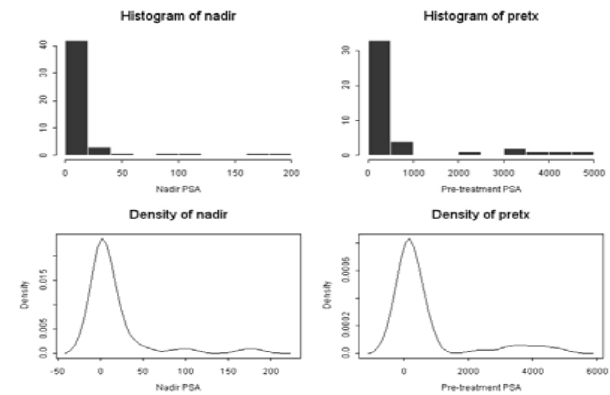
Ex: Categorizing Age

- g agetg = age
- recode agetg min/60=1 60/70=2 70/80=3 80/max=4
- tabulate agetg

agetg	Freq.	Percent	Cum.
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3	12	24.00	96.00
4	2	4.00	100.00
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Example: PSA Data Variables



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Example: PSA Data Variables

